INTERSECTING RATIONAL BEATTY SEQUENCES

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Received: 3/5/12, Revised: 3/24/13, Accepted: 7/6/13, Published: 8/12/13

Abstract
A rational Beatty sequence has the form \{[pi/q + b] : i \in \mathbb{Z}\} where p > q > 0 and gcd(p, q) = 1. We call p/q the modulus of the sequence and b the offset. Morikawa gave a condition on the moduli of two Beatty sequences such that they would be disjoint for a suitable choice of offsets. Holzman and Fraenkel showed that the sequence formed by the intersection of two Beatty sequences with moduli \(p_1/q_1\) and \(p_2/q_2\) could have as many as \(q_2 + 3\) distinct consecutive differences. In this note we show that if the moduli satisfy the Morikawa condition but the sequences do intersect then the consecutive differences take on at most three different values.

1. Introduction
A Beatty sequence has the form \{[i\alpha + \beta] : i \in \mathbb{Z}\}. We call \(\alpha\) the modulus and \(\beta\) the offset of the sequence. The sequences were named for Samuel Beatty [1] who asked for a proof that two Beatty sequences, with offsets equal to 0 and moduli \(\alpha_1\) and \(\alpha_2\), partition the positive integers if both moduli are irrational and \(1/\alpha_1 + 1/\alpha_2 = 1\). A proof of this pleasing result appeared in [2]. A Beatty sequence is rational or irrational according to whether its modulus is rational or irrational. Covering properties of irrational Beatty sequences are now well understood. See, for instance, [8] and its bibliography. This is not so for coverings by collections of rational Beatty sequences which are the subject of this paper. We write \(S(p/q, b)\), where gcd\((p, q) = 1\), for the Beatty Sequence \{\(pi/q + b\) : i \in \mathbb{Z}\}. We will assume throughout that \(b\) here is an integer – this involves no loss of generality by a result in [9]. The following famous conjecture is due to Aviezri Fraenkel [4].

**Conjecture 1.** If the collection of Beatty sequences \(\{S(p_i/q_i, b_i) : i = 1, \ldots, t\}\) partitions the integers with \(t > 2\) then \(\{p_1/q_1, \ldots, p_t/q_t\} = \{(2^t - 1)/2^{t-i} : 1 \leq i \leq t\}\).

This conjecture has generated a considerable literature. The strongest result to
date is by Bark and Varjú [3] who showed that any counterexample must have \( t > 7 \). See also the surveys [8], and Section F14 of [6].

A Beatty sequence may be regarded as an approximation to an arithmetic progression in that its consecutive differences take two values \( \{ \alpha \} \) and \( \{ \alpha' \} \) rather than one. The intersection properties of arithmetic progressions are given by the Chinese Remainder Theorem which we give, in a long-winded way, here.

**Theorem 2.** (Chinese Remainder Theorem) Let \( a_1, a_2, b_1, b_2 \) be integers with \( a_1 \) and \( a_2 \) positive.  
(a) There exist integers \( b_1 \) and \( b_2 \) such that \( S(a_1, b_1) \) and \( S(a_2, b_2) \) are disjoint if and only if \( \gcd(a_1, a_2) > 1 \).  
(b) If \( \gcd(a_1, a_2) = 1 \), then the intersection of \( S(a_1, b_1) \) and \( S(a_2, b_2) \) is an arithmetic progression with common difference \( a_1 a_2 \).  
(c) If \( \gcd(a_1, a_2) > 1 \), and \( S(a_1, b_1) \) and \( S(a_2, b_2) \) do intersect, then their intersection is an arithmetic progression with common difference \( \text{lcm}(a_1 a_2) \).

The situation for Beatty sequences is more complicated. Instead of part (a) we have the following result of Ryozu Morikawa [7], [10].

**Theorem 3.** (Japanese Remainder Theorem) With \( p = (p_1, p_2) \), \( q = (q_1, q_2) \), \( q_1 = u_1 q \) and \( q_2 = u_2 q \), there exist numbers \( b_1 \) and \( b_2 \) such that \( S(p_1/q_1, b_1) \) and \( S(p_2/q_2, b_2) \) are disjoint if and only if there exist positive integers \( x \) and \( y \) such that

\[
x u_1 + y u_2 = p - 2 u_1 u_2 (q - 1).
\]

When this is so we say that \( p_1/q_1 \) and \( p_2/q_2 \) satisfy the Morikawa condition.

**Definition 4.** If \( a_1, \ldots, a_n \) is an increasing sequence of integers then we say that the differences \( a_{i+1} - a_i \) are the gap sizes of the sequence. If \( S \) is a set of residues modulo \( p \), whose members have been reduced modulo \( p \) to integers in the interval \( [0, p-1] \), and labeled \( g_1 \leq g_2 \leq \cdots \leq g_n \), then the set of gap sizes of \( S \) is \( \{ g_{i+1} - g_i : i = 1, \ldots, n-1 \} \cup \{ p + g_1 - g_n \} \).

Instead of part (b) of Theorem 2 we have the following result of Fraenkel and Holzman [5].

**Theorem 5.** If \( S(p_1/q_1, b_1) \) and \( S(p_2/q_2, b_2) \) are Beatty sequences whose moduli do not satisfy the Morikawa condition, and \( q_1 \geq q_2 \geq 2 \), then the intersection of the two sequences has at most \( q_2 + 3 \) distinct gap sizes.

The bound here is best possible. In this paper we obtain an analogy of part (c) of Theorem 2 by giving a precise description of the intersection of two Beatty sequences whose moduli satisfy the Morikawa condition. In particular, it follows that in this case the intersection has at most three gap sizes.
2. Results

Notation 6. Throughout this section we will use the following notation. We will be considering Beatty sequences $S(p_1/q_1, b_1)$ and $S(p_2/q_2, b_2)$. We assume, without loss of generality, that $q_1 \leq q_2$. We put $p = \gcd(p_1, p_2)$, $p_1 = mp$, and $p_2 = np$. This implies

$$\gcd(m, q_1) = \gcd(n, q_2) = 1.$$ 

We set $\overline{q}_1$ and $\overline{q}_2$ to be the least non-negative residues satisfying $q_1 \overline{q}_1 \equiv -1 \pmod{p}$ and $q_2 \overline{q}_2 \equiv -1 \pmod{p}$ respectively. Similarly, $\overline{q}_m$ and $\overline{q}_n$ are the least non-negative residues satisfying $q_1 \overline{q}_m \equiv -1 \pmod{m}$ and $q_2 \overline{q}_n \equiv -1 \pmod{n}$, respectively. We set $k_1 = (q_1 \overline{q}_1 + 1)/p$ and $k_2 = (q_2 \overline{q}_2 + 1)/p$.

The argument proceeds in three steps. In Theorem 11 we obtain an expression for the intersection of $S(p/q_1, b_1)$ and $S(p/q_2, b_2)$. This is used in Theorem 15 to obtain an expression for the intersection of $S(pm/q_1, b_1)$ and $S(p/q_2, b_2)$, and that result is used in Theorem 16 to obtain an expression for the intersection of $S(pm/q_1, b_1)$ and $S(pm/q_2, b_2)$.

Definition 7. Let $b$, $n$, $p$, $d$ be positive integers with $n \leq p$, $\gcd(p, d) = 1$ and $S = \{id + b \mod p : i = 0, \ldots, n - 1\}$. Reduce each member of $S$ to an integer in $[0, p - 1]$ and label them $g_1, \ldots, g_n$, such that $g_1 \leq g_2 \leq \cdots \leq g_n$. We say that this sequence is a modular arithmetic progression modulo $p$ with additive difference $d$.

The following is easily derived from the usual Three Gap Theorem, see [11].

Theorem 8 (Three Gap Theorem). The set of gap sizes of a modular arithmetic progression has cardinality at most 3, and if the cardinality equals 3 then the largest member of the set equals the sum of the other two.

The following corollary follows immediately from the preceding theorem and Definition 7.

Corollary 9. If $g_1, \ldots, g_n$ is a modular arithmetic progression modulo $p$, then the set of gap sizes in the doubly infinite increasing sequence with range $\{g_i + jp : 1 \leq i \leq n, j \in \mathbb{Z}\}$ has cardinality at most 3, and if the cardinality equals 3 then the largest member of the set equals the sum of the other two.

The Beatty sequence $S = S(p/q, b)$ has period $p$ in the sense that $a \in S$ if and only if $a + p \in S$, and so is characterised by a set of residues modulo $p$. The following is Theorem 3 of [10].

Theorem 10. The sequence $S(p/q, b)$ with $\gcd(p, q) = 1$ coincides with the set of integers congruent modulo $p$ to a member of $\{iq + b : 0 \leq i \leq q - 1\}$, where $q \overline{q} \equiv -1 \pmod{p}$.
Thus the set of residues in a Beatty sequence forms a modular arithmetic progression, and the Beatty sequence itself fulfills the conditions of Corollary 9. In fact the Beatty sequence has at most two gap sizes. These are \([p/q]\) and \([\hat{p}/q]\) (which are equal when \(q = 1\)).

**Theorem 11.** Let \(p, q_1\) and \(q_2\) be positive integers with \(\gcd(p, q_1) = \gcd(p, q_2) = 1\) such that \(p/q_1\) and \(p/q_2\) satisfy the Morikawa condition. If \(b_1, b_2\) are integers such that \(S(p/q_1, b_1)\) and \(S(p/q_2, b_2)\) intersect then the intersection is the set of residues

\[
\{a \bar{q}_1 + iG_1 \bar{q}_1 + b_1 : 0 \leq i < t - 1 \}
\]

modulo \(p\) for some positive integer \(t\) where \(G_1\) is the smallest gap size in \(\{ - q_1 i \bar{q}_2 - q_1 b_2 \mod p : i = 0, \ldots, q_2 - 1 \}\) if \(t > 2\), and the second or third smallest gap size if \(t = 2\), and \(a\) is a non-negative integer satisfying

\[
a + G_1(t - 1) < q_1.
\]

The ideas of the following proof are illustrated in the accompanying figure.

**Proof.** Without loss of generality suppose \(b_1 = 0\). Let \(B_1\) be the set of residues modulo \(p\) in \(S(p/q_1, b_1)\) and \(B_2\) be the set in \(S(p/q_2, b_2)\). Theorem 10 implies that

\[
B_1 \equiv \{ i \bar{q}_1 : i = 0, \ldots, q_1 - 1 \} \mod p
\]

and

\[
B_2 \equiv \{ i \bar{q}_2 + b_2 : i = 0 \ldots q_2 - 1 \} \mod p.
\]

Let

\[
B_1^* \equiv \{ - q_1 i \bar{q}_1 : 0 \leq i \leq q_1 - 1 \} \mod p
\]

\[
\equiv \{ 0, \ldots, q_1 - 1 \} \mod p,
\]

and

\[
B_2^* \equiv \{ - q_1 i \bar{q}_2 - q_1 b_2 \mod p : 0 \leq i \leq q_2 - 1 \} \mod p.
\]

Clearly

\[
- q_1(B_1 \cap B_2) \equiv B_1^* \cap B_2^* \mod p
\]

so \(|B_1 \cap B_2| = |B_1^* \cap B_2^*|\). If \(|B_1 \cap B_2| = 1\) then we have nothing to prove, and if \(|B_1 \cap B_2| = 2\) then a simpler version of the proof applies (but note the comments at the end of the proof), so we assume \(|B_1 \cap B_2| \geq 3\). Consider the set of gaps in the modular arithmetic progression \(B_2^*\). By Theorem 8 there are at most 3 gap sizes. We will assume there are 3 (if there are less, then a simpler version of the proof applies) and that the gaps are \(G_1 < G_2 < G_3\). Since the moduli of the Beatty sequences satisfy the Morikawa condition there will be some value for \(b_2\) which makes \(|B_1 \cap B_2|\) empty, and therefore \(B_1^* \cap B_2^* = \{ 0, \ldots, q_1 - 1 \} \cap B_2^*\) empty, which implies \(G_3 > q_1\). Now consider a different value of \(b_2\) for which the sequences do intersect (note that this doesn’t change the gap sizes of \(B_2^*\) and let the intersection be the sequence

\[
0 \leq a_1 < a_2 < \cdots < a_t \leq q_1 - 1
\]
where, by the assumption above, \( t \geq 3 \). We claim that the only gap size in this sequence is \( G_1 \). Clearly no gap can equal \( G_3 \) since

\[
G_3 > q_1 > a_t - a_1. \tag{6}
\]

Suppose \( G_2 \) occurs in the sequence. Then, since \( t \geq 3 \), there is an adjacent gap of size at least \( G_1 \). This implies \( a_t - a_1 \geq G_1 + G_2 \), but \( G_1 + G_2 = G_3 \) by Theorem 8 and we get a contradiction as in (6). Thus all gaps equal \( G_1 \) and

\[
B_1^* \cap B_2^* = \{ a + iG_1 : 0 \leq i \leq t-1 \}
\]

for some integer \( a \). By (5) we have

\[
a + (t-1)G_1 \leq q_1 - 1. \tag{7}
\]

Therefore from (4)

\[
B_1 \cap B_2 \equiv \overline{q}_1 \{ a + iG_1 : 0 \leq i \leq t-1 \} \pmod{p}
\]

\[
\equiv \{ a\overline{q}_1 + iG_1\overline{q}_1, 0 \leq i \leq t-1 \} \pmod{p}
\]

which is (2). If there are only two elements in the intersection we cannot conclude that \( a_2 - a_1 < G_2 \). This observation leads to the anomalous case in the theorem. □

This completes the analysis of the case when two Beatty sequences with the same numerator in their moduli intersect. Before progressing to the more general case we prove the following theorem and its corollary.
Theorem 12. Let $m$ and $t$ be positive integers, and $a_1$, $a_2$, $b_1$ and $b_2$ be integers in the interval $[0, t - 1]$. If

$$\{a_1i + b_1 : i = 0, \ldots, t - 1\} \equiv \{a_2i + b_2 : i = 0, \ldots, t - 1\} \pmod{m} \quad (8)$$

and

$$(t + 1) \gcd(a_1, a_2, m) < m \quad (9)$$

then either $a_1 \equiv a_2 \pmod{m}$ and $b_1 \equiv b_2 \pmod{m}$, or $a_1 \equiv -a_2 \pmod{m}$ and $b_2 \equiv b_1 + a_1(t - 1) \pmod{m}$.

Proof. Let $S = \{a_1i + b_1 : i = 0, \ldots, t - 1\}$. We first show that neither $a_1t + b_1$ nor $a_1(t + 1) + b_1$ is congruent modulo $m$ to a member of $S$. Suppose otherwise. If $a_1t + b_1$ is congruent modulo $m$ to a member of $S$, then

$$a_1t + b_1 \equiv a_1i + b_1 \pmod{m} \text{ for some } i \in [0, t - 1]$$

$$\Rightarrow a_1(t - i) \equiv 0 \pmod{m},$$

which implies $i = 0$ else the members of $S$ would not be distinct. Hence $m$ divides $a_1t$ but $m$ does not divide $a_1i$ for any $i \in [1, t - 1]$. Hence $t$ divides $m$ and $m/t$ divides $a_1$. Thus $a_1 = Am/t$ for some integer $A$ where $\gcd(A, m) = 1$ and

$$S \equiv \{(Am/t)i + b_1 : i = 0, \ldots, t - 1\} \pmod{m}.$$ 

Clearly

$$\{(Am/t)i + b_1 : i = 0, \ldots, t - 1\} \equiv \{(m/t)i + b_1 : i = 0, \ldots, t - 1\} \pmod{m}$$

so $S \equiv \{mi/t + b_1 : i = 0, \ldots, t - 1\} \pmod{m}$. It follows that $m/t$ divides $a_2$. In fact we have

$$m \mid \gcd(a_1, m)t \text{ and } m \mid \gcd(a_2, m)t,$$

and thus $m$ divides $\gcd(a_1, a_2, m)t$ which implies $t \gcd(a_1, a_2, m) \geq m$, contradicting (9). We conclude $a_1t + b_1$ is not congruent modulo $m$ to any member of $S$.

Now suppose $a_1(t + 1) + b_1$ is congruent modulo $m$ to a member of $S$. As above this leads to

$$a_1(t + 1 - i) \equiv 0 \pmod{m}$$

for some $i$ in $[0, t - 1]$. In order for the members of $S$ to be distinct this implies $i = 0$ or $i = 1$. If $i = 1$ we get $a_1t \equiv 0 \pmod{m}$ which is impossible as in the previous case. Using similar reasoning to the previous case we see that if $i = 0$ which implies that $m$ divides $a_1(t + 1)$, $t + 1$ divides $m$, and

$$S \cup \{a_1t + b_1\} \equiv \{mi/(t + 1) + b_1 : i = 0, \ldots, t\} \pmod{m}.$$ 

Then $m/(t + 1)$ divides $a_2$ which leads to

$$m \mid \gcd(a_1, a_2, m)(t + 1),$$
implying that \((t + 1) \gcd(a_1, a_2, m) \geq m\), again contradicting (9). We conclude that 
\(a_1(t + 1) + b_1\) is not congruent modulo \(m\) to any member of \(S\). 

By similar reasoning we conclude that neither \(a_2 + b_2\) nor \(a_2(t + 1) + b_2\) is 
congruent modulo \(m\) to a member of \(S\). 

Now consider the set \(S' = \{a_2i + b_2 : i = 1, \ldots, t\}\) modulo \(m\). That is, \(S'\) 
is formed by adding \(a_2\) to each member of \(S\). Note that \(|S \cap S'| = t - 1\) since 
\(a_2(t + 1) + b_2\) is not congruent modulo \(m\) to any member of \(S\). Hence 
\[S \cap S' = \{a_1i + b_1 : i = 0, \ldots, t - 1, i \neq j\},\] 
for some \(j \in [0, t - 1]\). We will show that \(j\) equals 0 or \(t - 1\). Suppose, for the sake 
of contradiction, that \(0 < j < t - 1\). Since \(j > 0\), \(a_1(j - 1) + b_1\) belongs to \(S \cap S'\). 
Then, 
\[a_1(j - 1) + b_1 \equiv a_1(t - 1) + b_1 + a_2 \pmod{m},\] 
since if \(t - 1\) were replaced by \(k\) with \(0 \leq k < t - 1\), then \(a_1j + b_1\) would be congruent 
modulo \(m\) to \(a_1(k + 1) + b_1 + a_2\) and so belong to \(S'\). From (11) we therefore get 
\[a_2 \equiv (j - t)a_1 \pmod{m}.\] 
Now from our assumption that \(j < t - 1\) we have \(a_1(j + 1) + b_1\) congruent to a member 
of \(S\), and by (11) congruent to \(a_1(t + 1) + b_1 + a_2\) modulo \(m\). So \(a_1(t + 1) + b_1 + a_2\) 
is congruent modulo \(m\) to a member of \(S'\), which implies that \(a_1(t + 1) + b_1\) is 
congruent to a member of \(S\) which we showed earlier to be impossible. We conclude 
that \(j = 0\) or \(j = t - 1\). 

If \(j = 0\) then (10) gives 
\[S' \cap S \equiv \{a_1i + b_1 : i = 1, \ldots, t - 1\} \pmod{m},\] 
so that no member is congruent modulo \(m\) to \(b_1\), and \(a_1 + b_1 \equiv a_1k + b_1 + a_2\) for 
some \(k\) in \(\{0, \ldots, t - 1\}\). We must have \(k = 0\), else \(S' \cap S\) would include an element 
congruent to \(a_1(k - 1) + b_1 + a_2 \equiv b_1 \pmod{m}\). Hence we get \(a_1 \equiv a_2 \pmod{m}\), 
and from (8) we see that \(b_1 = b_2\). 

Similarly, if \(j = t - 1\) then (10) gives \(a_1(t - 2) + b_1 \equiv a_1k + b_1 + a_2 \pmod{m}\) 
for some \(k\) in \(\{0, \ldots, t - 1\}\), and this \(k\) must equal \(t - 1\) else \(a_1(t - 1) + b_1\) would 
be congruent modulo \(m\) to a member of \(S'\) and we get \(a_1 \equiv -a_2 \pmod{m}\). In this 
case (8) then gives 
\[
\{a_1i + b_1 : 0 \leq i \leq t - 1\}
\equiv \{a_1i + b_2 : 0 \leq i \leq t - 1\} \pmod{m}
\equiv \{a_1(t - 1 - i) + b_2 - a_1(t - 1) : 0 \leq i \leq t - 1\} \pmod{m}
\equiv \{a_1j + b_2 - a_1(t - 1) : 0 \leq j \leq t - 1\} \pmod{m},
\]
which implies 
\[b_1 \equiv b_2 - a_1(t - 1) \pmod{m},\]
Corollary 13. Using Notation 6 we have either

\[ a_2 \bar{q}_2 + b_2 \equiv a\bar{q}_1 + b_1 \pmod{p} \]

or

\[ a_2 \bar{q}_2 + b_2 \equiv \bar{q}_1(a + (q_2 - 1)G_1) + b_1 \pmod{p}. \]

Proof. Let \( H \) be the smallest gap size in \( \{-q_2 i\bar{q}_1 - q_2 b_1 \mod p : 0 \leq i \leq q_1 - 1\} \). By swapping the roles of \( S(p/q_1, b_1) \) and \( S(p/q_2, b_2) \) in Theorem 11 we can rewrite (2) as follows. The set of residues modulo \( p \) in the intersection of \( S(p/q_1, b_1) \) and \( S(p/q_2, b_2) \) is

\[ \{a_2 \bar{q}_2 + iH\bar{q}_2 + b_2 : 0 \leq i \leq t - 1\}, \]

where \( a_2 \) satisfies \( 0 \leq a_2 + H(t-1) \leq q_2 \). Note that we don’t need a different \( t \) value as the size of the intersection doesn’t change. We thus have, using the notation of the theorem,

\[ \{a\bar{q}_1 + iG_1\bar{q}_1 + b_1 : 0 \leq i \leq t - 1\} \equiv \{a_2 \bar{q}_2 + iH\bar{q}_2 + b_2 : 0 \leq i \leq t - 1\} \tag{12} \]

modulo \( p \). We will show that either \( H\bar{q}_2 \equiv G_1\bar{q}_1 \) or \( H\bar{q}_2 \equiv -G_1\bar{q}_1 \) modulo \( p \). This is immediate if \( t = 1 \) or \( t = 2 \), so we assume \( t \geq 3 \). In Notation 6 we assumed that \( q_1 \leq q_2 \). Since the moduli of our Beatty sequences satisfy the Morikawa condition (so the sequences would be disjoint for suitable offsets) we must have \( q_1 + q_2 \leq p \), and so \( q_1 \leq p/2 \). Then from (7) \( (t-1)G_1 < q_1 \leq p/2 \), so that, for \( t \geq 3 \),

\[ G_1(t + 1) < p. \]

We now apply the theorem with (12) in the role of (8). Since \( \bar{q}_1 \) and \( \bar{q}_2 \) are relatively prime to \( p \),

\[ \gcd(G_1\bar{q}_1, H\bar{q}_2, p) = \gcd(G_1, H, p) \leq G_1. \]

So with (13) we have \( \gcd(G_1\bar{q}_1, H\bar{q}_2, p)(t + 1) < p \), which plays the role of (9). We conclude that either \( H\bar{q}_2 \equiv G_1\bar{q}_1 \) and

\[ a_2 \bar{q}_2 + b_2 \equiv a\bar{q}_1 + b_1 \pmod{p}, \]

or \( H\bar{q}_2 \equiv -G_1\bar{q}_1 \) modulo \( p \) and

\[
\begin{align*}
 a_2 \bar{q}_2 + b_2 & \equiv a_1 \bar{q}_1 + b_1 + G_1 \bar{q}_1(t - 1) \pmod{p} \\
 & \equiv \bar{q}_1(a_1 + G_1(t - 1)) + b_1 \pmod{p},
\end{align*}
\]

as required. \( \square \)
We now analyse the intersection $S(p_1/q_1, b_1) \cap S(p/q_2, b_2)$.

**Lemma 14.** The set of residues modulo $pm$ in $S(pm/q_1, b)$ is

$$\{i(\overline{q}_1 + p\overline{q}_m k_1) + b_1 : 0 \leq j \leq q_1 - 1\}.$$  

**Proof.** By Theorem 10 the set of residues modulo $pm$ in $S(pm/q_1, b_1)$ is $\{i\overline{Q} + b_1 : 0 \leq i \leq q_1 - 1\}$, where $\overline{Q}$ is the least non-negative residue modulo $pm$ satisfying $q_1\overline{Q} \equiv -1 \pmod{pm}$. Using Notation 6 we then have $\overline{Q} \equiv \overline{q}_1 \pmod{p}$ so that

$$\overline{Q} q_1 = (\overline{q}_1 + lp)q_1 = -1 + k_1 p + l p q_1.$$  

But $\overline{Q} q_1 \equiv -1 \pmod{pm}$ so $k_1 + l q_1 \equiv 0 \pmod{m}$, which implies that $t \equiv \overline{q}_m k_1 \pmod{m}$, and the result follows. \hfill \Box

**Theorem 15.** We use Notation 6, recalling that $p_1 = pm$. If $p_1/q_1$ and $p/q_2$ satisfy the Morikawa condition then $S(p_1/q_1, b_1) \cap S(p/q_2, b_2)$ equals

$$\{(a + i G_1)(\overline{q}_1 + p\overline{q}_m k_1) + b_1 + \mu mp : 0 \leq i \leq t - 1, \mu \in \mathbb{Z}\},$$

where $a$, $G_1$ and $t$ have the same meaning as in Theorem 11.

**Proof.** We write $S_1$, $S_2$, and $S_m$ for $S(p/q_1, b_1)$, $S(p/q_2, b_2)$, and $S(pm/q_1, b_1)$ respectively. Since $S_m \subseteq S_1$ we have

$$S_m \cap S_2 = (S_1 \cap S_2) \cap S_m.$$  

By Theorem 11

$$S_1 \cap S_2 \equiv \{a \overline{q}_1 + i G_1 \overline{q}_1 + b_1 : 0 \leq i \leq t - 1\} \pmod{p}$$

where $a$ and $t$ are positive integers satisfying $0 < a + G_1(t - 1) \leq q_1$. By Lemma 14

$$S_m \equiv \{j(\overline{q}_1 + p\overline{q}_m k_1) + b_1 : 0 \leq j \leq q_1 - 1\} \pmod{pm}.$$  

Suppose $x \in S_2 \cap S_m$. Since $x \in S_1 \cap S_2$ we have

$$x = a \overline{q}_1 + i_1 G_1 \overline{q}_1 + b_1 + lp,$$

for some $i_1$ in $\{0, \ldots, t - 1\}$ and $l \in \mathbb{Z}$. Then, since $x \in S_m$,

$$a \overline{q}_1 + i_1 G_1 \overline{q}_1 + b_1 + lp = j_1(\overline{q}_1 + p\overline{q}_m k_1) + b_1 + \mu pm \tag{14}$$

for some $\mu \in \mathbb{Z}$ and $j_1$ in $\{0, \ldots, q_1\}$. It follows that

$$(a + i_1 G_1) \overline{q}_1 \equiv j_1 \overline{q}_1 \pmod{p}.$$
Since $0 < a + G_1(t - 1) \leq q_1 < p$ and $\gcd(q_1, p) = 1$, we have $a + i_1G_1 = j_1$. Then (14) gives

$$l = (a + i_1G_1)\overline{q}_m k_1 + \mu m.$$ 

The implications here can be reversed, so that $S_m \cap S_2$ equals

$$\{(a + iG_1)(\overline{q}_1 + p\overline{q}_m k_1) + b_1 + \mu mp : 0 \leq i \leq t - 1, \mu \in \mathbb{Z}\},$$

as required.

Now we obtain our main result.

**Theorem 16.** We use Notation 6 recalling that $p_1 = pm$ and $p_2 = pn$. If $p_1/q_1$ and $p_2/q_2$ satisfy the Morikawa condition then the intersection of the Beatty sequences $S_m = S(pm/q_1, b_1)$ and $S_n = S(pm/q_2, b_2)$ is the modular arithmetic progression given below, where $a$, $G_1$, and $t$ have the same meaning as in Theorem 11:

$$\{a(\overline{q}_1 + pk_1\overline{q}_m) + b_1 + \mu_0 mp + i(G_1(\overline{q}_1 + pk_1\overline{q}_m) + \mu_1 mp) : 0 \leq i \leq t - 1\}$$

modulo $mnp$, and one of the following cases holds.

**Case 1** In this case $\mu_0$ is the least non-negative residue satisfying

$$\mu_0 m \equiv \frac{a_2\overline{q}_2 + b_2 - a_1\overline{q}_1 - b_1}{p} + a_2k_2\overline{q}_n - a_1k_1\overline{q}_m \pmod{n},$$

and $\mu_1$ is the least non-negative residue satisfying

$$\mu_1 m \equiv \frac{G_2\overline{q}_2 - G_1\overline{q}_1}{p} + G_2k_2\overline{q}_n - G_1k_1\overline{q}_m \pmod{n}.$$ 

**Case 2** In this case $\mu_0$ is the least non-negative residue satisfying

$$\mu_0 m \equiv \frac{a_1\overline{q}_1 + b_1 - a_2\overline{q}_2 - b_2}{p} + G_2(\overline{q}_2 + pk_2\overline{q}_n) - a_1k_1\overline{q}_m + a_2k_2\overline{q}_n \pmod{n},$$

and $\mu_1$ is the least non-negative residue satisfying

$$\mu_1 m \equiv \frac{-G_1\overline{q}_1 + G_2\overline{q}_2}{p} - G_1k_1\overline{q}_m - G_2k_2\overline{q}_n \pmod{n}.$$ 

**Proof.** By Theorem 15 $S_m \cap S_2$ equals

$$\{(a_1 + iG_1)(\overline{q}_1 + pk_1\overline{q}_m) + b_1 + \mu mp : 0 \leq i \leq t - 1, \mu \in \mathbb{Z}\},$$

(15)

and $S_n \cap S_1$ equals

$$\{(a_2 + iG_2)(\overline{q}_2 + pk_2\overline{q}_n) + b_2 + \nu mp : 0 \leq i \leq t - 1, \nu \in \mathbb{Z}\}.$$ 

(16)
Since $S_m \subseteq S_1$ and $S_n \subseteq S_2$ we can obtain $S_m \cap S_n$ by evaluating $(S_m \cap S_2) \cap (S_n \cap S_1)$. Suppose $x \in S_m \cap S_n$. Then there exist integers $i_1, i_2 \in \{1, \ldots, t\}$ and $\mu, \nu \in \mathbb{Z}$ such that

$$x = i_1 G_1(\overline{q}_1 + pk_1\overline{q}_m) + a_1(\overline{q}_1 + pk_1\overline{q}_m) + b_1 + \mu mp$$

$$= i_2 G_2(\overline{q}_2 + pk_2\overline{q}_n) + a_2(\overline{q}_2 + pk_2\overline{q}_n) + b_2 + \nu np. \quad (17)$$

Considering this modulo $p$ we get

$$i_1 G_1\overline{q}_1 + a_1\overline{q}_1 + b_1 \equiv i_2 G_2\overline{q}_2 + a_2\overline{q}_2 + b_2 \pmod{p}. \quad (18)$$

This is the congruence considered in Corollary 13. We therefore have either $i_2 = i_1$ or $i_2 = p - i_1$. We suppose the first of these holds and return to the other case at the end of the proof. Then (17) gives

$$i_1(G_1(\overline{q}_1 + pk_1\overline{q}_m) - G_2(\overline{q}_2 + pk_2\overline{q}_n)) + a_1(\overline{q}_1 + pk_1\overline{q}_m)$$

$$- a_2(\overline{q}_2 + pk_2\overline{q}_n) + b_1 - b_2 = \nu np - \mu mp$$

for some integers $\mu$ and $\nu$. Using Corollary 13 again we may divide through by $p$ getting

$$\frac{i_1(G_1\overline{q}_1 - G_2\overline{q}_2 + G_1k_1\overline{q}_m - G_2k_2\overline{q}_n)}{p} + \frac{a_1\overline{q}_1 + b_1 - a_2\overline{q}_2 - b_2}{p}$$

$$+ a_1k_1\overline{q}_m - a_2k_2\overline{q}_n = \nu n - \mu m. \quad (19)$$

Since $m$ and $n$ are relatively prime, we can find $\mu$ and $\nu$ satisfying this for any $i_1$. Consider the case $i_1 = 0$, and let $\mu_0$ and $\nu_0$ be the unique values of $\mu \in \{0, \ldots, n-1\}$ and $\nu \in \{0, \ldots, m-1\}$ satisfying,

$$\nu_0 n - \mu_0 m \equiv \frac{a_1\overline{q}_1 + b_1 - a_2\overline{q}_2 - b_2}{p} + a_1k_1\overline{q}_m - a_2k_2\overline{q}_n \pmod{mn}, \quad (20)$$

and let $\mu_1$ and $\nu_1$ be the unique values of $\mu \in \{0, \ldots, n-1\}$ and $\nu \in \{0, \ldots, m-1\}$ satisfying,

$$\nu_1 n - \mu_1 m \equiv \frac{G_1\overline{q}_1 - G_2\overline{q}_2}{p} + G_1k_1\overline{q}_m - G_2k_2\overline{q}_n \pmod{mn}. \quad (21)$$

We see that (19) will be satisfied for any $i_1 \in \{0, \ldots, t-1\}$ if we set

$$\mu = \mu_0 + i_1\mu_1 + sn$$

$$\nu = \nu_0 + i_1\nu_1 + sm,$$

where $s$ is any integer. Substituting in (15) and replacing $i_1$ with $i$ we get,

$$S_m \cap S_n = \{(a_1 + iG_1)(\overline{q}_1 + pk_1\overline{q}_m) + b_1 + (\mu_0 + i\mu_1 + sn)mp$$

$$: 0 \leq i \leq t - 1, s \in \mathbb{Z}\}$$

$$= \{(a_1\overline{q}_1 + pk_1\overline{q}_m) + b_1 + \mu_0 mp + i(G_1(\overline{q}_1 + pk_1\overline{q}_m) + \mu_1 mp)$$

$$+ smnp : 0 \leq i \leq t - 1, s \in \mathbb{Z}\},$$
as required.
If we put \( i_2 = p - i_1 \) instead of \( i_2 = i_1 \) as a consequence of (18) then (17) gives the values of \( \mu_0 \) and \( \mu_1 \) in Case 2. \( \square \)

**Corollary 17.** If \( p_1/q_1 \) and \( p_2/q_2 \) satisfy the Morikawa condition then the intersection of \( S_1 = S(pm/q_1, b_1) \) and \( S_2 = S(pm/q_2, b_2) \) contains at most 3 distinct gap sizes.

**Proof.** Immediate from Theorem 16 and Corollary 9. \( \square \)

We end with an example. Consider the pair of Beatty Sequences \( S(737/10, 0) \), and \( S(469/15, 2) \). This gives \( p = 67, m = 11, n = 7, q_1 = 10, q_2 = 15, b_1 = 0, b_2 = 2, \eta_1 = 20, \eta_2 = 58, \eta_m = 1, k_1 = 3, k_2 = 13, G_1 = 2, G_2 = 3, \) and \( t = 4 \). By Theorem 11 the intersection of \( S(737/10, 0) \) and \( S(469/15, 2) \) is \( \{20a_1 + 40i : 0 \leq i \leq t - 1\} \mod 67 \). With \( a_1 = 3 \) this gives the intersection \( \{6, 33, 46, 6\} \). We similarly have \( a_2 = 1 \). Now consider Theorem 16. We have \( \mu_0 \) being the least non-negative residue satisfying

\[
\mu_0 m \equiv \frac{a_2 \eta_2 + b_2 - a_1 \eta_1 - b_1}{p} + a_2 k_2 \eta_n - a_1 k_1 \eta_m \quad \text{(mod } n) \]

\[\implies 11 \mu_0 \equiv 69 \quad \text{(mod } 7)\]

\[\implies \mu_0 \equiv 5 \quad \text{(mod } 7),\]

and \( \mu_1 \) is the least non-negative residue satisfying

\[
\mu_1 m \equiv \frac{G_2 \eta_2 - G_1 \eta_1}{p} + G_2 k_2 \eta_n - G_1 k_1 \eta_m \quad \text{(mod } n) \]

\[\implies 11 \mu_1 \equiv 230 \quad \text{(mod } 7)\]

\[\implies \mu_1 \equiv 5 \quad \text{(mod } 7).\]

Then \( S_m \cap S_n \) is

\[
\{a_1 (\eta_1 + pk_1 \eta_m) + b_1 + \mu_0 mp + i(G_1 \eta_1 + pk_1 \eta_m) + \mu_1 mp : 0 \leq i \leq t - 1\} \mod pmn \]

\[\equiv \{4384 + 4127i : 0 \leq i \leq 3\} \mod 5159 \]

\[\equiv \{1252, 2284, 3316, 4384\} \mod 5159.\]

**Acknowledgements:** Thanks to Julie Caddy, Aviezri Fraenkel, Greg Gamble and Amy Glen for advice and assistance, and to the referee for very careful reading.
References


