Pricing Volatility Swaps Under Heston’s Stochastic
Volatility Model with Regime Switching

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ABSTRACT

We develop a model for pricing volatility derivatives, such as variance swaps and
volatility swaps under a continuous-time Markov-modulated version of the stochastic

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volatility (SV) model developed by Heston (1993). In particular, we suppose the parameters of our version of Heston’s SV model depend on the states of a continuous-time observable Markov chain process, which can be interpreted as the states of an observable macroeconomic factor. The market we consider is incomplete in general, and hence, there is more than one equivalent martingale pricing measure. We adopt the regime switching Esscher transform used by Elliott et al. (2005) to determine a martingale pricing measure for the valuation of variance and volatility swaps in this incomplete market. We consider both the probabilistic approach and the partial differential equation, (PDE), approach for the valuation of volatility derivatives.

Key words: Regime Switching Esscher Transform; Markov-modulated Heston’s SV model; Observable Markov Chain Process; Volatility Swaps; Variance Swaps; Regime Switching OU-process
§1. Introduction and Summary

Volatility is one of the major features used to describe and measure the fluctuations of asset prices. It is popular as a measure of risk and uncertainty. It plays a significant role in three pillars of modern financial analysis: risk management, option valuation and asset allocation. There are different measures of volatility, including realised volatility, implied volatility and model-based volatility. Realised volatility is the standard deviation of the historical financial returns; implied volatility is the volatility inferred from the market option price data based on an assumed option pricing model, such as the Black-Scholes model; model-based volatility includes both parametric and non-parametric specifications of the volatility dynamics, such as the ARCH models introduced by Engle (1982), the GARCH models introduced by Bollerslev (1986) and Taylor (1986), independently, and their variants. These models are introduced to provide a better specification, measurement and forecasting of volatilities of various financial assets. Recent major financial issues, such as the collapse of LTCM, the Asian financial crisis and the problems of Barings and Orange Country, reveal that the global financial markets have become more volatile. Due to large and frequent shifts in the volatilities of various assets in the recent past, there has been a growing and practical need to develop some models with related financial instruments to hedge volatility risk. On the other hand, market speculators may be interested in
guessing the direction that the volatility may take in the future. These practical needs facilitate the growth of the market for derivative products related to volatility. The initial suggestions for such products included options and futures on the volatility index. Brenner and Galai (1989, 1993) proposed the idea of developing a volatility index. The Chicago Board Options Exchange (CBOE) introduced a volatility index in 1993, called VIX, which was based on implied volatilities from options on the S&P 500 index. The VIX Index reflects the market expectations of near-term volatility inferred from the S&P 500 stock index option prices. For a detailed discussion about various volatility derivative products, see Brenner et al. (2001).

Variance swaps and volatility swaps are popular volatility derivative products and they have been actively traded in over-the-counter markets since the collapse of LTCM in late 1998. In particular, the variance swaps and the volatility swaps on stock indices, currencies and commodities are quoted and traded actively. These products are popular among market practitioners as a hedge for volatility risk. The variance swap is a forward contract in which the long position pays a fixed amount $K_{\text{var}}$/1 nominal value at the maturity date and receives the floating amount $(\sigma^2)_R$/1 nominal value, where $K_{\text{var}}$ is the strike price and $(\sigma^2)_R$ is the realized variance. The volatility swap is the same as the variance swap, except that the realized variance $(\sigma^2)_R$ is replaced by the realized volatility $(\sigma)_R$. 
variance swaps in the context of a fractional Black-Scholes market. Swishchuk (2005) developed a model for valuing variance swaps under a stochastic volatility model with delay.

In this paper, we develop a model for pricing volatility derivatives, such as variance swaps and volatility swaps under a continuous-time Markov-modulated version of Heston’s stochastic volatility (SV) model, (Heston (1993)). In particular, the parameters of a continuous-time version of Heston’s SV model depend on the states of a continuous-time observable Markov chain process, which can be interpreted as the states of an observable macroeconomic factor, such as an observable economic indicator for business cycles and the sovereign ratings for the region. The market described by the Markov-modulated model is incomplete in general, and hence, there is more than one equivalent martingale pricing measure. We adopt the regime switching Esscher transform introduced by Elliott et al. (2005) to determine a martingale pricing measure for the valuation of variance and volatility swaps. We consider both the probabilistic approach and the PDE approach for the valuation of volatility derivatives. We assume that the Markov chain process is observable in both the probabilistic approach and the PDE approach. We shall document economic consequences for the prices of the variance swaps and volatility swaps of the incorporation of the regime-switching in the volatility dynamics by conducting a Monte Carlo experiment.
The next section describes the model. Section three demonstrates the use of the probabilistic approach for pricing volatility and variance swaps under the continuous-time Markov-modulated version of Heston’s stochastic volatility model. Section four considers the use of the PDE approach for the valuation. In Section five, we shall conduct the Monte Carlo experiment for comparing the prices implied by the stochastic volatility models with and without switching regimes. We shall document the economic consequences for the prices of switching regimes. The final section suggests some possible topics for further investigation.

§2. The Model

In this section, we adopt the probabilistic approach for the valuation of volatility derivatives under a continuous-time Markov-modulated stochastic volatility model. Our model can be considered the regime-switching augmentation of the model by Swishchuk (2004) for pricing and hedging volatility swaps. The Markov-modulated version of Heston’s stochastic volatility model can describe the consequences for the asset price and volatility dynamics of the transitions of the states of an observable macroeconomic factor, which affects the asset prices and volatility dynamics, such as an observable economic indicator of business cycles, or the sovereign ratings of the region by some international rating agencies, such as
the Standard & Poors, Moodys and Fitch, etc. In particular, the parameters in the asset price dynamics and the stochastic volatility dynamics depend on the state of an economic indicator, which is described by the observable Markov chain.

First, consider a continuous-time financial model with two primary securities, namely a risk-free bond $B$ and a risky stock $S$. Fix a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with $\mathcal{P}$ being the real-world probability measure. Let $T$ be the time index set $[0, \infty)$. On $(\Omega, \mathcal{F}, \mathcal{P})$ we consider a continuous-time finite-state observable Markov chain $X := \{X_t\}_{t \in T}$ with state space $S$ which might be the set $\{s_1, s_2, \ldots, s_N\}$, where $s_i \in \mathbb{R}^N$, for $i = 1, 2, \ldots, N$. The states of the Markov chain process $X$ describe the states of an observable economic indicator. Without loss of generality, the state space of the chain can be identified with the set $\{e_1, e_2, \ldots, e_N\}$ of unit vectors in $\mathbb{R}^N$.

Write $\Pi(t)$ for the generator, or $Q$-matrix, $[\pi_{ij}(t)]_{i,j=1,2,\ldots,N}$ of $X$. Following Elliott et al. (1994), the following semi-martingale representation theorem for the process $X$ can be obtained:

$$X_t = X_0 + \int_0^t \Pi(s)X_s ds + M_t.$$ (2.1)

Here $\{M_t\}_{t \in T}$ is an $\mathcal{R}^N$-valued martingale increment process with respect to the natural filtration generated by $X$.

Let $W^1 := \{W^1_t\}_{t \in T}$ and $W^2 := \{W^2_t\}_{t \in T}$ denote two correlated standard
Brownian Motions on \((\Omega, \mathcal{F}, \mathcal{P})\) with respect to the \(\mathcal{P}\)-augmentation of the filtration \(\mathcal{F}^W := \{\mathcal{F}^W_t\}_{t \in T}\), where \(W := (W^1, W^2)\). We assume that

\[
\text{Cov}(dW^1_t, dW^2_t) = \rho \, dt ,
\]  

(2.2)

where \(\rho \in (-1, 1)\) is the instantaneous correlation coefficient between \(W^1\) and \(W^2\).

We further suppose that the processes \(X\) is independent with \((W^1, W^2)\). Pan (2002) documented from her empirical studies on stock indices that the correlation coefficient between the diffusive shocks to the volatility level and the level of the underlying price is significantly negative. The model considered here can incorporate this negative correlation coefficient.

Let \(\{r(t, X_t)\}_{t \in T}\) be the instantaneous market interest rate of the bond \(B\), which depends on the state of the economic indicator described by \(X\); that is,

\[
r(t, X_t) = < r, X_t > , \ t \in T ,
\]  

(2.3)

where \(r := (r_1, r_2, \ldots, r_N)\) with \(r_i > 0\) for each \(i = 1, 2, \ldots, N\) and \(< \cdot, \cdot >\) denotes the inner product in \(\mathcal{R}^N\).

To simplify the notation, write \(r_t\) for \(r(t, X_t)\). Then, the dynamics of the price
process \( \{ B_t \}_{t \in T} \) for the bond \( B \) are described by:

\[
 dB_t = r_t B_t dt , \quad B_0 = 1 .
\] (2.4)

Suppose the expected appreciation rate \( \{ \mu_t \}_{t \in T} \) of the risky stock \( S \) depends on the state of the economic indicator \( X \) and is described by:

\[
 \mu_t := \mu(t, X_t) = \langle \mu, X_t \rangle , \quad \text{where} \quad \mu := (\mu_1, \mu_2, \ldots, \mu_N) \text{ with } \mu_i \in \mathcal{R}, \text{ for each } i = 1, 2, \ldots, N .
\] (2.5)

Let \( \{ \alpha_t \}_{t \in T} \) denote the long-term volatility level. We assume that \( \alpha_t \) depends on the state of the economic indicator \( X \) and is given by:

\[
 \alpha_t := \alpha(t, X_t) = \langle \alpha, X_t \rangle , \quad \text{where} \quad \alpha := (\alpha_1, \alpha_2, \ldots, \alpha_N) \text{ with } \alpha_i > 0, \text{ for each } i = 1, 2, \ldots, N .
\] (2.6)

Let \( \beta \) and \( \gamma \) denote the speed of mean reversion and the volatility of volatility, respectively. (In general, we could consider a more general case where both the speed of mean reversion \( \beta \) and the volatility of volatility \( \gamma \) depend on the states of the economic indicator \( X \).) However, in order to make our model more analytically tractable, we suppose that both are constant. Suppose that the dynamics of the price
process $\{S_t\}_{t \in T}$ and the short-term volatility process $\sigma := \{\sigma_t\}_{t \in T}$ of the risky stock are governed by the following equations:

$$
\begin{align*}
    dS_t &= \mu_t S_t dt + \sigma_t S_t dW_t^1 , \\
    d\sigma_t^2 &= \beta(\alpha_t^2 - \sigma_t^2) dt + \gamma \sigma_t dW_t^2 , \\
    \end{align*}
$$

(2.7)

where $\text{Cov}(dW_t^1, dW_t^2) = \rho dt$.

Note that the variance process $\sigma_t^2$ follows a Cox, Ingersoll and Ross (1985) process.

Let $\tilde{\rho} := \sqrt{1-\rho}$; Write $W := \{W_t\}_{t \in T}$ for a standard Brownian motion, which is independent of $W^1$ and $X$. Then, we can write the dynamics of $\{S_t\}_{t \in T}$ and $\sigma := \{\sigma_t\}_{t \in T}$ as:

$$
\begin{align*}
    dS_t &= \mu_t S_t dt + \sigma_t S_t dW_t^1 , \\
    d\sigma_t^2 &= \beta(\alpha_t^2 - \sigma_t^2) dt + \tilde{\rho} \sigma_t dW_t^2 + \rho \gamma \sigma_t dW_t^1 . \\
    \end{align*}
$$

(2.8)

Let $Y_t$ denote the logarithmic return $\ln(S_t/S_0)$ over the interval $[0, t]$. Then,

$$
Y_t = \int_0^t \left( \mu_u - \frac{1}{2} \sigma_u^2 \right) du + \int_0^t \sigma_u dW_u^1 .
$$

(2.9)

In our model, there are three sources of randomness: $X$, $W^1$ and $W^2$. Let $\mathcal{F}^X := \{\mathcal{F}_t^X\}_{t \in T}$, $\mathcal{F}^{W_1} := \{\mathcal{F}_t^{W_1}\}_{t \in T}$ and $\mathcal{F}^{W_2} := \{\mathcal{F}_t^{W_2}\}_{t \in T}$ be the $\mathcal{P}$-augmentation of the natural filtrations generated by $\{X_t\}_{t \in T}$, $\{W_1^1\}_{t \in T}$ and $\{W_1^2\}_{t \in T}$, respectively. Let
\( \mathcal{F}^S := \{ \mathcal{F}^S_t \}_{t \in T} \) denote the \( \mathcal{P} \)-augmentation of the natural filtration generated by \( \{ S_t \}_{t \in T} \).

Our model is a regime switching version of Heston’s stochastic volatility model and is in general incomplete. There are, therefore, infinitely many equivalent martingale pricing measures. In the sequel, we shall adopt the regime-switching Esscher transform developed in Elliott et al. (2005) to determine an equivalent martingale pricing measure for the volatility and variance swaps. The regime-switching Esscher transform provides market practitioners with a convenient and flexible way to determine an equivalent martingale measure in the incomplete market. The choice of the equivalent martingale measure can be justified by the regime-switching minimal entropy equivalent martingale (MEMM) measure (See Elliott et al. (2005)). One drawback of using the Esscher transform is that the pricing rule by the Esscher transform is not linear, which is considered by financial economists as a desirable property for a pricing rule. There are other possible approaches for determining an equivalent martingale measure in an incomplete market, for instance, the minimum variance hedging in Duffie and Richardson (1991) and Schweizer (1992). In the minimum-variance hedging, the intrinsic value process corresponding to a given contingent claim is used as the optimal tracking
process. The intrinsic value process is defined by the minimal martingale measure introduced in Föllmer and Schweizer (1991) and Schweizer (1991) and corresponds to the risk-neutral approach for valuing the claim. The minimum-variance hedging can provide a pertinent solution to address the pricing and hedging of a contingent claim while the Esscher transform mainly deals with the valuation of the claim. The hedging strategies are optimal in sense of minimizing the expected quadratic costs. The Esscher transform can provide a convenient and intuitive way to price a claim.

Let $G_t$ denote the right continuous completion of the $\sigma$-algebra $\mathcal{F}_t^X \vee \mathcal{F}_t^{W^3} \vee \mathcal{F}_t^{W^2}$, for each $t \in T$. Let $\Theta_t := \Theta(t, X_t, \sigma_t)$ denote a regime switching Esscher process, which is written as follows:

$$\Theta_t = \Theta(t, X_t, \sigma_t) = \langle \Theta(t, \sigma_t), X_t \rangle,$$  \hspace{1cm} (2.10)

where $\Theta(t, \sigma_t) := (\Theta(t, \sigma_t, e_1), \Theta(t, \sigma_t, e_2), \ldots, \Theta(t, \sigma_t, e_N))$ and $\Theta(t, \sigma_t, e_i)$ is $\mathcal{F}_t^{W^2}$ measurable, for each $i = 1, 2, \ldots, N$. So, $\Theta(t, X_t, \sigma_t)$ is an $N$-dimensional $\mathcal{F}_t^X \vee \mathcal{F}_t^{W^2}$-measurable random vector.

Following Elliott et al. (2005), for such a process $\Theta$ we define the regime switching Esscher transform $Q_{\Theta} \sim P$ on $G_t$ with respect to a family of parameters $\{\Theta_u\}_{u \in [0, t]}$. 

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by:
\[
\frac{dQ_\Theta}{dP} \bigg|_{G_t} = \exp \left( \int_0^t \Theta_u dY_u \right) \exp \left( \int_0^t \Theta_u dY_u \right) \bigg| \mathcal{F}_t^X \vee \mathcal{F}_t^{W^2} \bigg) , \quad t \in T .
\] (2.11)

Note that \( \int_0^t \Theta_u dY_u | \mathcal{F}_t^X \vee \mathcal{F}_t^{W^2} \sim N(\int_0^t \Theta_u (\mu_u - \frac{1}{2}\sigma_u^2)du, \int_0^t \Theta_u^2 \sigma_u^2 du) \), that is a normal distribution with mean \( \int_0^t \Theta_u (\mu_u - \frac{1}{2}\sigma_u^2)du \) and variance \( \int_0^t \Theta_u^2 \sigma_u^2 du \), under \( P \). Then given \( \mathcal{F}_t^X \vee \mathcal{F}_t^{W^2} \), the Radon-Nikodym derivative of the regime switching Esscher transform is given by:
\[
\frac{dQ_\Theta}{dP} \bigg|_{G_t} = \exp \left( \int_0^t \Theta_u \sigma_u dW_u^1 - \frac{1}{2} \int_0^t \Theta_u^2 \sigma_u^2 du \right) .
\] (2.12)

The central tenet of the fundamental theorem of asset pricing, which is also called the fundamental theorem of finance in Ross (2005), established the equivalence between the absence of arbitrage opportunities and the existence of a positive linear pricing operator (or positive state space prices). It was first established by Ross (1973) in a finite state space setting. Cox and Ross (1976) provided the first statement of risk-neutral pricing. Ross (1978) and Harrison and Kreps (1979) then extended the fundamental theorem in a general probability space and characterized the risk-neutral pricing as the expectation of the discounted asset’s payoff with respect to an equivalent martingale measure. Harrison and Kreps (1979), and Harrison and Pliska (1981, 1983) established the equivalence between the absence of arbitrage and the existence of an equivalent martingale measure under which all discounted price
processes are martingale. The fundamental theorem was then extended by several authors, including Dybvig and Ross (1987), Back and Pliska (1991), Schachermayer (1992) and Delbaen and Schachermayer (1994). In our setting, let \( \hat{\Theta} := \{\hat{\Theta}_t\}_{t \in T} \) denote a process for the risk-neutral regime switching Esscher parameters. Due to the presence of the uncertainty generated the processes \( X \) and \( W^2 \), the martingale condition is characterised by considering an enlarged filtration and requiring:

\[
S_0 = E_{Q_\Theta} \left[ \exp \left( -\int_0^t r_s ds \right) S_t \bigg| \mathcal{F}_t^X \vee \mathcal{F}_t^{W^2} \right] , \quad \text{for any } t \in T .
\]

(2.13)

One can interpret this condition as one when information about the Markov chain and the stochastic volatility process are known to the market’s agent in advance.

Give these arguments which are similar to those in Elliott et al. (2005), it can be shown that the martingale condition (2.11) implies that \( \hat{\Theta}_t := \hat{\Theta}(t, X_t, \sigma_t) \) should be given by

\[
\hat{\Theta}_t = \frac{r(t, X_t) - \mu(t, X_t)}{\sigma_t^2} = -\frac{\lambda(t, X_t, \sigma_t)}{\sigma_t} , \quad t \in T ,
\]

(2.14)

where \( \lambda_t := \lambda(t, X_t, \sigma_t) \in \mathcal{G}_t \) is the market price of risk at time \( t \).

Then \( \hat{\Theta}_t = \langle \hat{\Theta}(t, \sigma_t), X_t \rangle \), where \( \hat{\Theta}(t, \sigma_t) = (\frac{r_1-\mu_1}{\sigma_t}, \frac{r_2-\mu_2}{\sigma_t}, \ldots, \frac{r_N-\mu_N}{\sigma_t}) \). This is an \( N \)-dimensional \( \mathcal{F}_t^{W^2} \)-measurable random vector.
Using (2.12), the Radon-Nikodym derivative of $\mathcal{Q}_{\tilde{\Theta}}$ with respect to $\mathcal{P}$ is given by:

$$
\left. \frac{d\mathcal{Q}_{\tilde{\Theta}}}{d\mathcal{P}} \right|_{\mathcal{G}_t} = \exp\left[ \int_0^t \left( \frac{r_u - \mu_u}{\sigma_u} \right) dW_u^1 - \frac{1}{2} \int_0^t \left( \frac{r_u - \mu_u}{\sigma_u} \right)^2 du \right].
$$

(2.15)

By Girsanov’s theorem, $\tilde{W}_1^1 = W_1^1 + \int_0^t (r_s - \mu_s) ds$ is a standard Brownian motion with respect to $\{\mathcal{G}_t\}_{t \in T}$ under $\mathcal{Q}_{\tilde{\Theta}}$. Since $W$ and $X$ are independent of $W^1$, $W$ is a standard Brownian motion under $\mathcal{Q}_{\tilde{\Theta}}$ and $X$ remains unchanged under the change of the probability measure from $\mathcal{P}$ to $\mathcal{Q}_{\tilde{\Theta}}$. Let $\tilde{\alpha}^2_t := \alpha^2_t - \rho \gamma (r_t - \mu_t)$. Then, the dynamics of $S$ and $\sigma$ under $\mathcal{Q}_{\tilde{\Theta}}$ are

$$
dS_t = r_t S_t dt + \sigma_t S_t d\tilde{W}_1^1,
$$

$$
d\sigma^2_t = \beta (\tilde{\alpha}^2_t - \sigma^2_t) dt + \rho \gamma \sigma_t d\tilde{W}_1^1 + \rho \gamma \sigma_t dW_t.
$$

(2.16)

Let $\tilde{W}_1^2 := \rho \tilde{W}_1^1 + \rho W_t$. Then, the dynamics of $\sigma$ can be written as:

$$
d\sigma^2_t = \beta (\tilde{\alpha}^2_t - \sigma^2_t) dt + \gamma \sigma_t d\tilde{W}_2^1.
$$

(2.17)

When there is no regime switching, (i.e. the Markov chain $X$ is degenerate), the risk-neutral dynamics under $\mathcal{Q}_{\tilde{\Theta}}$ reduce to the risk-neutral dynamics in Heston (1993).

§3. The Probabilistic Approach

In this section, we assume the dynamics of the long-term volatility level $\{\alpha_t\}_{t \in T}$ switch over time according to one of the regimes determined by the state of an observable Markov chain. We may interpret the states of the observable Markov chain as those of
some observable economic indicator. First, we shall consider the valuation of variance swaps, which is more simple than the valuation of volatility swaps. **Then, we shall discuss the hedging of variance swaps and volatility swaps.**

§3.1. Valuation

A variance swap is a forward contract on annualized variance, which is the square of the realized annual volatility. Let $\sigma^2_R(S)$ denote the realized annual stock variance over the life of the contract. Then,

$$\sigma^2_R(S) := \frac{1}{T} \int_0^T \sigma^2_u du.$$  \hfill (3.1)

In practice, variance swaps are written on the realized variance evaluated based on daily closing prices with the integral in (3.1) replaced by a discrete sum. Hence, variance swaps with payoffs depending on the realized variance defined in (3.1) are only approximations to those of the actual contracts. See Javaheri et al. (2002) for discussions on this point.

Let $K_v$ and $N$ denote the delivery price for variance and the notional amount of the swap in dollars per annualized volatility point squared. Then, the payoff of the variance swap at expiration time $T$ is given by $N(\sigma^2_R(S) - K_v)$. Intuitively, the buyer of the variance swap will receive $N$ dollars for each point by which the realized annual variance $\sigma^2_R(S)$ has exceeded the variance delivery price $K_v$. We can adopt the
risk-neutral regime switching Esscher transform $\mathcal{Q}_\theta$ for the valuation of the variance swap. In fact, the value of the variance swap can be evaluated as the expectation of its discounted payoff with respect to the measure $\mathcal{Q}_\theta$, which is exactly the same as the value of a forward contract on future realized variance with strike price $K_v$.

As in Elliott, Sick and Stein (2003), we initially consider the evaluation of the conditional value, or price, of a derivative given the information about the sample path of the Markov chain process from time 0 to time $T$, say $\mathcal{F}_T^X$. In particular, given $\mathcal{F}_T^X$, the conditional price of the variance swap $P(X)$ is given by:

$$P(X) = E_{\mathcal{Q}_\theta} [e^{-\int_0^T r_u du} N(\sigma_R^2(S) - K_v)|\mathcal{F}_T^X]$$

$$= e^{-\int_0^T r_u du} NE_{\mathcal{Q}_\theta} (\sigma_R^2(S)|\mathcal{F}_T^X) - e^{-\int_0^T r_u du} NK_v. \quad (3.2)$$

Hence, the valuation of the variance swap given $\mathcal{F}_T^X$ can be reduced to the problem of calculating the mean value of the underlying variance $E_{\mathcal{Q}_\theta} (\sigma_R^2(S)|\mathcal{F}_T^X)$.

Note that under $\mathcal{Q}_\theta$, the volatility dynamics can be written as:

$$\sigma_t^2 = \sigma_0^2 + \int_0^t \beta (\tilde{\alpha}_s^2 - \sigma_s^2)ds + \int_0^t \gamma \sigma_s d\tilde{W}_s. \quad (3.3)$$

Given $\mathcal{F}_T^X$, $\tilde{\alpha}_t^2$ is a known function of time $t$. Hence,

$$E_{\mathcal{Q}_\theta} (\sigma_t^2|\mathcal{F}_T^X) = \sigma_0^2 + \int_0^t \beta [\tilde{\alpha}_s^2 - E_{\mathcal{Q}_\theta} (\sigma_s^2|\mathcal{F}_T^X)]ds. \quad (3.4)$$

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This implies that

\[
\frac{dE_{Q_\alpha}(\sigma_t^2|\mathcal{F}_T^X)}{dt} = \beta [\hat{\alpha}_t^2 - E_{Q_\alpha}(\sigma_t^2|\mathcal{F}_T^X)].
\] (3.5)

Solving (3.5) gives:

\[
E_{Q_\alpha}(\sigma_t^2|\mathcal{F}_T^X) = \sigma_0^2 e^{-\beta t} + \beta \int_0^t \hat{\alpha}_s^2 e^{-\beta (t-s)} ds.
\] (3.6)

By Itô’s lemma,

\[
\sigma_t^4 = \sigma_0^4 + \int_0^t [2\beta \sigma_s^2 (\hat{\alpha}_s^2 - \sigma_s^2) + \gamma^2 \sigma_s^2] ds + \int_0^t 2\gamma \sigma_s^3 d\tilde{W}_s.
\] (3.7)

Hence,

\[
\frac{dE_{Q_\alpha}(\sigma_t^4|\mathcal{F}_T^X)}{dt} = (2\beta \hat{\alpha}_t^2 + \gamma^2) E_{Q_\alpha}(\sigma_t^2|\mathcal{F}_T^X) - 2\beta E_{Q_\alpha}(\sigma_t^4|\mathcal{F}_T^X).
\] (3.8)

Solving (3.8) gives:

\[
E_{Q_\alpha}(\sigma_t^4|\mathcal{F}_T^X) = \sigma_0^4 e^{-2\beta t} + \int_0^t e^{-2\beta (t-s)} (2\beta \hat{\alpha}_s^2 + \gamma^2) \left( \sigma_0^2 e^{-\beta s} + \beta \int_0^s \hat{\alpha}_u^2 e^{-\beta (s-u)} du \right) ds.
\] (3.9)

Hence,

\[
\text{Var}_{Q_\alpha}(\sigma_t^2|\mathcal{F}_T^X) = E_{Q_\alpha}(\sigma_t^4|\mathcal{F}_T^X) - E_{Q_\alpha}^2(\sigma_t^2|\mathcal{F}_T^X)
\]

\[
= \sigma_0^2 \frac{\gamma^2}{\beta} (e^{-\beta t} - e^{-2\beta t}) + \int_0^t \left[ e^{-2\beta (t-s)} (2\beta \hat{\alpha}_s^2 + \beta \gamma^2) \int_0^s \hat{\alpha}_u^2 e^{-\beta (s-u)} du \right] ds
\]

\[-\beta^2 \left( \int_0^t \hat{\alpha}_s^2 e^{-\beta (t-s)} ds \right)^2.
\] (3.10)
The results for $E_{\mathcal{Q}_t}(\sigma^2_t|\mathcal{F}^X_T)$ and $\text{Var}_{\mathcal{Q}_t}(\sigma^2_t|\mathcal{F}^X_T)$ are consistent with those in Shreve (2004).

Write $V := \sigma^2_R(S)$ to simplify the notation. Then, $E_{\mathcal{Q}_t}(V|\mathcal{F}^X_T)$ can be calculated as follows:

$$E_{\mathcal{Q}_t}(V|\mathcal{F}^X_T) = \frac{1}{T} \int_0^T \left( \sigma_0^2 e^{-\beta t} + \beta \int_0^t \tilde{\alpha}_s^2 e^{-\beta(t-s)} ds \right) dt$$

$$= \frac{\sigma_0^2}{\beta T} (1 - e^{-\beta T}) + \frac{\beta}{T} \int_0^T \left( \int_0^t \tilde{\alpha}_s^2 e^{-\beta(t-s)} ds \right) dt , \quad (3.11)$$

We evaluate $\text{Var}_{\mathcal{Q}_t}(V|\mathcal{F}^X_T)$ as follows:

$$\text{Var}_{\mathcal{Q}_t}(V|\mathcal{F}^X_T) = Cov_{\mathcal{Q}_t}(V, V|\mathcal{F}^X_T)$$

$$= 1/T^2 \int_0^T \int_0^T Cov_{\mathcal{Q}_t}(\sigma^2_t, \sigma^2_s|\mathcal{F}^X_T) dt ds \quad (3.12)$$

We first derive an expression for $Cov_{\mathcal{Q}_t}(\sigma^2_t, \sigma^2_s|\mathcal{F}^X_T)$. Without loss of generality, we suppose that $t > s$. Then, we define $\eta_{t,s}$ as follows:

$$\eta_{t,s} := \sigma^2_t - E_{\mathcal{Q}_t}(\sigma^2_s|\mathcal{F}^W_s \vee \mathcal{F}^X_T)$$

$$= \sigma^2_t - \sigma^2_s e^{-\beta(t-s)} - \beta \int_s^t \tilde{\alpha}_u^2 e^{-\beta(t-u)} du . \quad (3.13)$$

Then, it is immediate that

$$E_{\mathcal{Q}_t}(\eta_{t,s}|\mathcal{F}^W_s \vee \mathcal{F}^X_T) = 0 , \quad (3.14)$$
\[ E_{\tilde{\Theta}_t}(\eta_{t,s}|F_T) = 0, \quad (3.15) \]

so

\[ \text{Cov}_{\tilde{\Theta}_t}(\eta_{t,s}, \sigma_s^2|F_T) = 0. \quad (3.16) \]

Hence,

\[
\text{Cov}_{\tilde{\Theta}_t}(\sigma_t^2, \sigma_s^2|F_T) = \text{Cov}_{\tilde{\Theta}_t}(\eta_{t,s} + \sigma_s^2 e^{-\beta(t-s)} + \beta \int_s^t \tilde{\alpha}_u^2 e^{-\beta(t-u)} du, \sigma_s^2|F_T)
\]

\[
= e^{-\beta(t-s)} \text{Var}_{\tilde{\Theta}_t}(\sigma_s^2|F_T)
\]

\[
= e^{-\beta(t-s)} \left\{ \frac{\sigma_0^2 \gamma^2}{\beta} (e^{-\beta s} - e^{-2\beta s}) + \int_0^s \left[ e^{-2\beta(s-u)} \left( 2\beta^2 \tilde{\alpha}_u^2 + \beta \gamma^2 \right) \right] du - \beta^2 \left( \int_0^s \tilde{\alpha}_u^2 e^{-\beta(s-u)} du \right)^2 \right\}.
\quad (3.17)
\]

Therefore,

\[
\text{Var}_{\tilde{\Theta}_t}(V|F_T) = 1/T^2 \int_0^T \int_0^T e^{-\beta(t-s)} \left\{ \frac{\sigma_0^2 \gamma^2}{\beta} (e^{-\beta s} - e^{-2\beta s}) + \int_0^s \left[ e^{-2\beta(s-u)} \left( 2\beta^2 \tilde{\alpha}_u^2 + \beta \gamma^2 \right) \right] du - \beta^2 \left( \int_0^s \tilde{\alpha}_u^2 e^{-\beta(s-u)} du \right)^2 \right\} dtds
\]

\[
= \frac{\sigma_0^2 \gamma^2}{\beta^2 T^2} \left[ (1 - e^{-\beta T}) T + \frac{1}{4\beta} (1 - e^{-2\beta T})^2 \right] + 1/T^2 \int_0^T \int_0^T e^{-\beta(t-s)} \left\{ \int_0^s \left[ e^{-2\beta(s-u)} \left( 2\beta^2 \tilde{\alpha}_u^2 + \beta \gamma^2 \right) \right] du - \beta^2 \left( \int_0^s \tilde{\alpha}_u^2 e^{-\beta(s-u)} du \right)^2 \right\} dtds.
\quad (3.18)
\]

For each \( i = 1, 2, \ldots, N \), let \( \tilde{\alpha}_i := \alpha_i - \rho \gamma (r_i - \mu_i) \); Write \( \tilde{\alpha} := (\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_N) \). Given
\( \mathcal{F}_T^X \), the conditional price of the variance swap is given by:

\[
P(X) = e^{-\int_0^T r_u du} N \left[ \frac{\sigma_0^2}{\beta_T} (1 - e^{-\beta T}) + \frac{\beta}{T} \int_0^T \left( \int_0^t \beta \left< \dot{\alpha}^2, X_t \right> e^{-\beta(t-s)} ds \right) dt - K_v \right] \tag{3.19}
\]

Consider now the valuation of the volatility swap given \( \mathcal{F}_T^X \). A stock volatility swap is a forward contract on the annualized volatility. Let \( K_s \) denote the annualized volatility delivery price and \( N \) is the notational amount of the swap in dollar per annualized volatility point. Then, the payoff function of the volatility swap is given by \( N(\sigma_R(S) - K_s) \), where \( \sigma_R(S) := \sqrt{\frac{1}{T} \int_0^T \sigma_u^2 du} \). In other words, the payoff of the volatility swap is equal to the payoff of the variance swap when \( \sigma_R^2(S) \) is replaced by \( \sigma_R(S) \) and \( K_v \) is replaced by \( K_s \). Given \( \mathcal{F}_T^X \), the conditional price of the volatility swap is given by:

\[
P_s(X) = E_{\tilde{Q}_\Theta} [e^{-\int_0^T r_u du} N(\sigma_R(S) - K_s)] \big| \mathcal{F}_T^X] \]

\[
= e^{-\int_0^T r_u du} NE_{\tilde{Q}_\Theta}(\sigma_R(S) | \mathcal{F}_T^X) - e^{-\int_0^T r_u du} NK_s
\]

\[
= e^{-\int_0^T r_u du} NE_{\tilde{Q}_\Theta}(\sqrt{\sigma} | \mathcal{F}_T^X) - e^{-\int_0^T r_u du} NK_s. \tag{3.20}
\]

For the valuation of the volatility swap, we need to evaluate \( E_{\tilde{Q}_\Theta}(\sqrt{\sigma} | \mathcal{F}_T^X) \). We adopt the approximation for \( E_{\tilde{Q}_\Theta}(\sqrt{\sigma} | \mathcal{F}_T^X) \) introduced by Brockhaus and Long (2000), based on the second-order Taylor expansion for the function \( \sqrt{\sigma} \). This approximation method has also been adopted in Javaheri et al. (2002) and Swishchuk (2004).
It gives
\[ E_{Q_0}(\sqrt{V}|\mathcal{F}_T^X) \approx \sqrt{E_{Q_0}(V|\mathcal{F}_T^X)} - \frac{Var_{Q_0}(V|\mathcal{F}_T^X)}{8[E_{Q_0}(V|\mathcal{F}_T^X)]^{3/2}}, \tag{3.21} \]
where the term \( \frac{Var_{Q_0}(V|\mathcal{F}_T^X)}{8[E_{Q_0}(V|\mathcal{F}_T^X)]^{3/2}} \) is the convexity adjustment.

Hence, given \( \mathcal{F}_T^X \), the conditional price of the volatility swap can be approximated as:
\[
P_s(X) \approx e^{-\int_0^T r_u \, du} N \left[ \sqrt{E_{Q_0}(V|\mathcal{F}_T^X)} - \frac{Var_{Q_0}(V|\mathcal{F}_T^X)}{8[E_{Q_0}(V|\mathcal{F}_T^X)]^{3/2}} - K_s \right]. \tag{3.22} \]
§3.2. Hedging

Hedging variance swaps and volatility swaps is a challenging but practically important task. Javaheri et al. (2002) contended that hedging these products is difficult in practice. Hence, they considered the pricing of a variance swap and a volatility swap by the expectations of the discounted payoffs under the real-world probability measure. This is an example of actuarial-based valuation method. In this section, we shall discuss the hedging of variance swaps and volatility swaps. Different methods on hedging variance swaps, have been proposed in the literature. These methods include the simple delta hedging, the delta-gamma hedging, hedging using option portfolios, hedging using a log contract and the vega hedging, etc. For a comprehensive overview of various hedging strategies, see Demeterfi et al. (1999), Howison et al. (2004) and Windcliff et al. (2003). Simple delta hedging does not work well since it is an implicit linear approximation, which cannot incorporate the effects of large price movements and the realized variance or volatility will increase substantially when the underlying share prices move either up or down dramatically. This is the case even one considers a Geometric Brownian Motion for the price dynamics of the underlying share. Simple delta hedging is even more difficult to apply when one considers a stochastic volatility model and a regime-switching stochastic volatility model, which is even more complicated. Hedging using option portfolios and hedging using a log contract works
well when one considers an asset price model with non-stochastic volatility. The vega hedging provides market practitioners with a convenient way to hedge variance swaps and volatility swaps under a stochastic volatility model, in particular, the Heston SV model (see Howison et al. (2004) and Carr (2005)). Howison et al. (2004) considered the use of Vega to hedge volatility derivatives and derived a general formula for the Vega of a volatility derivative. Here, following Howison et al. (2004), we adopt the Vega hedging for a variance swap and a volatility swap since it can provide a convenient way to hedge these products under the regime-switching Heston’s SV model. In the case of the volatility swap, we shall derive an approximate formula for the Vega of the contract based on the approximate price of the contract.

First, we consider the hedging of the variance swap. Let $I_t := \int_0^t \sigma_u^2 du$. Write $\mathcal{F}_t^{W^2}$ for the $\mathcal{P}$-augmentation of the $\sigma$-algebra generated by the values of $\tilde{W}^2$ up to and including time $t$. Note that given $\mathcal{F}_T^X, \mathcal{F}_t^{W^2}$ is equivalent to $\mathcal{F}_t^{W^2}$. Then, using the results in Section 3.1, the price of the variance swap $P(t, X)$ at time $t$ is given by:

$$
P(t, X) = \frac{1}{T} e^{-\int_t^T r_u du} N\left[I_t + E_{Q_\alpha} \left( \int_t^T \sigma_u^2 du \mathbb{1}_{\mathcal{F}_t^{W^2}} \mathcal{F}_t^{W^2} \right) - TK_v \right]
$$

$$
= \frac{1}{T} e^{-\int_t^T r_u du} N\left[I_t + \frac{\sigma^2}{\beta} (e^{-\beta t} - e^{-\beta T}) + \beta \int_t^T \left( \int_s^t \tilde{\alpha}_u^2, X_u > e^{-\beta(s-u)} du \right) ds - TK_v \right].
$$

(3.23)
Let $\hat{\sigma}_t^2 := \gamma \sigma_t^2$, which represents the instantaneous volatility of the variance process at time $t$. Then, the Vega of the variance swap is given by:

\[
\frac{\partial P(t, X)}{\partial \hat{\sigma}} = \frac{2N\hat{\sigma}_t}{T\beta \gamma^2} (e^{-\beta t} - e^{-\beta T}) = \frac{2N\sigma_t}{T\beta} (e^{-\beta t} - e^{-\beta T}),
\]

which can be evaluated given the current value of the volatility level $\sigma_t$.

We shall consider the hedging of the volatility swap. First, we define $R_1(t, \sigma_t^2)$, $R_2(t, \sigma_t^2)$ and $R_3(t, \sigma_t^2)$ as follows:

\[
R_1(t, \sigma_t^2) = I_t + \frac{\sigma_t^2}{\beta} (e^{-\beta t} - e^{-\beta T}) + \beta \int_t^T \left( \int_t^s < \hat{\sigma}^2, X_u > e^{-\beta (s-u)} du \right) ds,
\]

\[
R_2(t, \sigma_t^2) = \frac{\sigma_t^2 \gamma^2}{T^2 \beta^3} \left[ e^{-\beta (T-t)^2} - \frac{1}{\beta} e^{-2\beta (T-t)} \right] + \frac{1}{T^2} \int_t^T \int_t^T \left\{ e^{-\beta (h-s)} \int_0^s \left[ e^{-2\beta (s-u)} \left( 2\beta^2 \hat{\alpha}_u^2 + \beta \gamma^2 \right) \int_0^u \hat{\alpha}_z^2 e^{-\beta(u-z)} dz \right] du - \beta^2 \left( \int_0^s \hat{\alpha}_u^2 e^{-\beta (s-u)} du \right)^2 \right\} dhds,
\]

and

\[
R_3(t, \sigma_t^2) = \frac{\sigma_t^2 \gamma^2}{\beta^2 T^2} \left[ (1 - e^{-\beta t}) t + \frac{1}{4\beta} (1 - e^{-2\beta t})^2 \right] + 1/T^2 \int_0^t \int_0^t \left\{ e^{-\beta (u-s)} \int_0^s \left[ e^{-2\beta (s-u)} \left( 2\beta^2 \hat{\alpha}_u^2 + \beta \gamma^2 \right) \int_0^u \hat{\alpha}_z^2 e^{-\beta(u-z)} dz \right] du - \beta^2 \left( \int_0^s \hat{\alpha}_u^2 e^{-\beta (s-u)} du \right)^2 \right\} duds.
\]

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From the results in Section 3.1, the price of the volatility swap \( P_s(t, X) \) at time \( t \) can be approximated as:

\[
P_s(t, X) \approx e^{-\int_0^T r u d u} \left[ \sqrt{E_{Q^t}(V|\mathcal{F}_T^Y \vee \mathcal{F}_t^W)} - \frac{\text{Var}_{Q^t}(V|\mathcal{F}_T^Y \vee \mathcal{F}_t^W)}{8} - K_s \right]
\]

\[
= e^{-\int_0^T r u d u} \left[ \sqrt{R_1(t, \sigma_t^2)} - \frac{R_2(t, \sigma_t^2)}{8 R_1(t, \sigma_t^2)^{3/2}} - K_s \right]. \tag{3.25}
\]

Write \( R_i \) for \( R_i(t, \sigma_t^2) \) \((i = 1, 2, 3)\). Then, the Vega of the volatility swap is approximated as:

\[
\frac{\partial P_s(t, X)}{\partial \sigma} = e^{-\int_0^T r u d u} \left[ \frac{\sigma_t}{\beta} R_1^{-1/2}(e^{-\beta t} - e^{-\beta T}) + \frac{3\sigma_t}{8\beta}(R_3 + R_2)R_1^{-5/2} \right.
\]

\[
(e^{-\beta t} + e^{-\beta T}) + \frac{\sigma_t \gamma^2 R_1}{4T^2 \beta^3} \left( e^{-\beta(T-t)^2} + \frac{1}{\beta} e^{-2\beta(T-t)} \right), \tag{3.26}
\]

which can be evaluated given the current value of the volatility level \( \sigma_t \).

§4. The P.D.E. Approach

In this section, we adopt a partial differential equation (P.D.E.) approach for evaluating the expectations of the discounted values of \( V \) and \( V^2 \), which are useful for computing the prices of the variance swaps and volatility swaps. The P.D.E. approach has been adopted by Javaheri, et al. (2002) for the valuation and hedging of volatility swaps within the framework of a GARCH(1, 1) stochastic volatility model. Here, we provide a regime switching modification of the problem and derive regime switching P.D.E.s and the corresponding systems of coupled P.D.E.s satisfied by the
expectations of the discounted values of $V$ and $V^2$. We adopt a regime switching version of the Feynman-Kac formula to obtain the regime switching P.D.Es. The derivation of the regime switching version of the Feynman-Kac formula follows from the martingale approach and Itô’s differentiation rule in Buffington and Elliott (2002).

First, let $V_t := \sigma^2_{R,t}(S) := \frac{1}{T} \int_0^t \sigma^2_u du$. Then, given $\mathcal{F}_t^{W^2} \lor \mathcal{F}_{T}^X$, the price of the variance swap is given by:

$$P(X, t) = e^{-\int_t^T r_u du} NE_{\tilde{Q}_a}(\sigma^2_{R,T}(S)|\mathcal{F}_t^{W^2} \lor \mathcal{F}_T^X) - e^{-\int_t^T r_u du} NK_v ,$$  \hspace{1cm} (4.1)

and the price of the volatility swap is given by:

$$P_s(X, t)$$

$$= e^{-\int_t^T r_u du} NE_{\tilde{Q}_a}(\sigma_{R,T}(S)|\mathcal{F}_t^{W^2} \lor \mathcal{F}_T^X) - e^{-\int_t^T r_u du} NK_s .$$  \hspace{1cm} (4.2)

Now, suppose $\sigma_t^2 = \Sigma$, $X_t = X$ and $V_t = V$ are given at time $t$. Then, the price of the variance swap is given by:

$$P(X, \Sigma, V, t) = E_{\tilde{Q}_a}(P(X, t)|\sigma_t^2 = \Sigma, V_t = V, X_t = X) ,$$ \hspace{1cm} (4.3)

and the price of the volatility swap is given by:

$$P_s(X, \Sigma, V, t) = E_{\tilde{Q}_a}(P_s(X, t)|\sigma_t^2 = \Sigma, V_t = V, X_t = X) .$$ \hspace{1cm} (4.4)

Buffington and Elliott (2002a, b) adopted a similar method to determine the price of a standard European call option.
By the double expectation formula,

\[
P(X, \Sigma, V, t) = NE_{Q_\tilde{\Theta}} \left[ e^{-\int_t^T r_u du} \sigma_t^2 \sigma_{R,T}^2(S) - e^{-\int_t^T r_u du} K_v \sigma_t^2 = \Sigma, V_t = V, X_t = X \right], \quad (4.5)
\]

and

\[
P_s(X, \Sigma, V, t) = NE_{Q_\tilde{\Theta}} \left[ e^{-\int_t^T r_u du} \sigma_t^2 \sigma_{R,T}^2(S) - e^{-\int_t^T r_u du} K_v \sigma_t^2 = \Sigma, V_t = V, X_t = X \right]. \quad (4.6)
\]

Hence, for the evaluation of \( P(X, \Sigma, V, t) \) and \( P_s(X, \Sigma, V, t) \), we need to compute

1. \( E_{Q_\tilde{\Theta}} \left[ e^{-\int_t^T r_u du} \sigma_t^2 = \Sigma, V_t = V, X_t = X \right] \)
2. \( E_{Q_\tilde{\Theta}} \left[ e^{-\int_t^T r_u du} \sigma_{R,T}^2(S) \sigma_t^2 = \Sigma, V_t = V, X_t = X \right] \)
3. \( E_{Q_\tilde{\Theta}} \left[ e^{-\int_t^T r_u du} \sigma_{R,T}^2(S) \sigma_t^2 = \Sigma, V_t = V, X_t = X \right] \)

The first expectation is equal to the value of a zero-coupon bond at time \( t \) given that \( \sigma_t^2 = \Sigma, V_t = V, X_t = X \), which pays one unit of account at maturity time \( T \). It is given by:

\[
E_{Q_\tilde{\Theta}} \left[ e^{-\int_t^T r_u du} \sigma_t^2 = \Sigma, V_t = V, X_t = X \right] = E_{Q_\tilde{\Theta}} \left( e^{-\int_t^T r_u du} X_t = X \right) := B(t, T, X). \quad (4.7)
\]
Let $B := \text{diag} r - \Pi^*$, where $\text{diag} r$ is the matrix with the vector $r$ on its diagonal.

Write $\phi_t(r) := \exp[-B(T - t)]I$. Then, by Elliott and Kopp (2004), $B(t, T, X)$ is given by:

$$B(t, T, X) = \langle \phi_t(r), X \rangle = \langle \exp[-B(T - t)], X \rangle > I.$$  \hfill (4.8)

The third expectation can be approximated by using the formula in Section 2. More specifically,

$$E_{Q_{\tilde{\Theta}}} \left[ e^{-\int_0^T r u d u} \sigma_{R,T}^2(S) \middle| \sigma^2_t = \Sigma, V_t = V, X_t = X \right] \approx \sqrt{E_{Q_{\tilde{\Theta}}} \left[ e^{-2\int_0^T r u d u} \sigma_{R,T}^2(S) \middle| \sigma^2_t = \Sigma, V_t = V, X_t = X \right] - Var_{Q_{\tilde{\Theta}}} \left[ e^{-2\int_0^T r u d u} \sigma_{R,T}^2(S) \middle| \sigma^2_t = \Sigma, V_t = V, X_t = X \right]}.$$  \hfill (4.9)

Hence, in order to provide an approximation to the third expectation, we need to evaluate

1. $E_{Q_{\tilde{\Theta}}} (e^{-4\int_0^T r u d u} \sigma_{R,T}^4(S)) | \sigma^2_t = \Sigma, V_t = V, X_t = X)

2. $E_{Q_{\tilde{\Theta}}} (e^{-2\int_0^T r u d u} \sigma_{R,T}^2(S)) | \sigma^2_t = \Sigma, V_t = V, X_t = X$.

Suppose $\mathcal{H}_t := \sigma \{ W^2_u, X_u | u \in [0, t] \}$. Since $V_t$ is a path integral of $\sigma^2_t$ and $\sigma^2_t$ is a Markov process given knowledge of $X$, $(V_t, \sigma^2_t)$ is a two-dimensional Markov process.
process given the knowledge of \( X \). Since \( X \) is also a Markov process, \( (X_t, \sigma^2_t, V_t) \) is a three-dimensional Markov process with respect to the information set \( \mathcal{H}_t \). Hence,

\[
M_1(X, \Sigma, V, t) := E_{\mathcal{Q}_\theta}(e^{-\int_t^T r_u du} \sigma^2_{R,T}(S)|\sigma^2_t = \Sigma, V_t = V, X_t = X)
= E_{\mathcal{Q}_\theta}(e^{-\int_t^T r_u du} \sigma^2_{R,T}(S)|\mathcal{H}_t),
\]

(4.10)

\[
M_2(X, \Sigma, V, t) := E_{\mathcal{Q}_\theta}(e^{-2\int_t^T r_u du} \sigma^2_{R,T}(S)|\sigma^2_t = \Sigma, V_t = V, X_t = X)
= E_{\mathcal{Q}_\theta}(e^{-2\int_t^T r_u du} \sigma^2_{R,T}(S)|\mathcal{H}_t),
\]

(4.11)

and

\[
M_3(X, \Sigma, V, t) := E_{\mathcal{Q}_\theta}(e^{-4\int_t^T r_u du} \sigma^2_{R,T}(S)|\sigma^2_t = \Sigma, V_t = V, X_t = X)
= E_{\mathcal{Q}_\theta}(e^{-4\int_t^T r_u du} \sigma^2_{R,T}(S)|\mathcal{H}_t),
\]

(4.12)

Now, write

\[
\tilde{M}_1(X, \Sigma, V, t) := e^{-\int_0^t r_u du} M_1(X, \Sigma, V, t)
= E_{\mathcal{Q}_\theta}(e^{-\int_0^T r_u du} \sigma^2_{R,T}(S)|\mathcal{H}_t),
\]

(4.13)

\[
\tilde{M}_2(X, \Sigma, V, t) := e^{-2\int_0^t r_u du} M_2(X, \Sigma, V, t)
= E_{\mathcal{Q}_\theta}(e^{-2\int_0^T r_u du} \sigma^2_{R,T}(S)|\mathcal{H}_t),
\]

(4.14)

and

\[
\tilde{M}_3(X, \Sigma, V, t) := e^{-4\int_0^t r_u du} M_3(X, \Sigma, V, t)
= E_{\mathcal{Q}_\theta}(e^{-4\int_0^T r_u du} \sigma^2_{R,T}(S)|\mathcal{H}_t).
\]

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\[
E_{Q_{\tilde{\Theta}}}(e^{-\int_0^T ru\,du}\sigma_{R,T}^4(S)|\mathcal{H}_t) .
\] (4.15)

Then, it can be shown that \(\tilde{M}_1, \tilde{M}_2\) and \(\tilde{M}_3\) are \(\mathcal{H}_t\)-martingales under \(Q_{\tilde{\Theta}}\).

In the sequel, we shall derive the P.D.E. for \(\tilde{M}_1, \tilde{M}_2\) and \(\tilde{M}_3\). For each \(i = 1, 2, 3\), let \(\tilde{M}_i(\Sigma, V, t)\) denote the \(N\)-dimensional vector \((\tilde{M}_i(e_1, \Sigma, V, t), \ldots, \tilde{M}_i(e_N, \Sigma, V, t))\). Then,

\[
\tilde{M}_i(X, \Sigma, V, t) = \langle \tilde{M}_i(\Sigma, V, t), X_t \rangle .
\] (4.16)

Then, by applying Itô’s differentiation rule to \(\tilde{M}_i(X, \Sigma, V, t)\),

\[
\tilde{M}_i(X, \Sigma, V, t) = \tilde{M}_i(X, \Sigma, V, 0) + \int_0^t \left( \frac{\partial \tilde{M}_i}{\partial t} + \beta(\tilde{\alpha}_u^2 - \sigma_u^2)\frac{\partial \tilde{M}_i}{\partial \Sigma} + \frac{1}{2}\gamma^2 \sigma_u^2 \frac{\partial^2 \tilde{M}_i}{\partial \Sigma^2} \\
+ \sigma_u^2 \frac{\partial \tilde{M}_i}{\partial V} \right)du + \int_0^t \frac{\partial \tilde{M}_i}{\partial \Sigma} \gamma \sigma_u d\tilde{W}_u + \int_0^t \tilde{M}_i \, dX_u \right),
\] (4.17)

and

\[
dX_t = \Pi(t)X_t dt + dM_t .
\] (4.18)

Due to the fact that \(\tilde{M}_1, \tilde{M}_2\) and \(\tilde{M}_3\) are \(\mathcal{H}_t\)-martingale under \(Q_{\tilde{\Theta}}\), all terms with bounded variation in the above Itô’s integral representation for \(\tilde{M}_i \ (i = 1, 2, 3)\) must be identical to zero. Hence, for \(i = 1, 2, 3\), \(\tilde{M}_i\) satisfies the following P.D.E.:

\[
\frac{\partial \tilde{M}_i}{\partial t} + \beta(\tilde{\alpha}_i^2 - \sigma_i^2)\frac{\partial \tilde{M}_i}{\partial \Sigma} + \frac{1}{2}\gamma^2 \sigma_i^2 \frac{\partial^2 \tilde{M}_i}{\partial \Sigma^2} + \sigma_i^2 \frac{\partial \tilde{M}_i}{\partial V} + \tilde{M}_i, \Pi X = 0 .
\] (4.19)
Now, let $M_i(\Sigma, V, t)$ denote the $N$-dimensional vector $(M_i(e_1, \Sigma, V, t), \ldots, M_i(e_N, \Sigma, V, t))$, where $i = 1, 2, 3$. Then,

$$M_i(X, \Sigma, V, t) = \langle M_i(\Sigma, V, t), X \rangle.$$  \hspace{1cm} (4.20)

$M_1$ satisfies the following P.D.E.:

$$\left( \exp \left( - \int_0^t r_u du \right) \right) \left( - r_t M_1 + \frac{\partial M_1}{\partial t} + \beta(\tilde{\alpha}_i^2 - \sigma_i^2) \frac{\partial M_1}{\partial \Sigma} + \frac{1}{2} \gamma^2 \sigma_i^2 \frac{\partial^2 M_1}{\partial \Sigma^2} + \sigma_i^2 \frac{\partial M_1}{\partial V} + \langle M_1, \Pi X \rangle \right) = 0,$$

with terminal condition $M_1(X, \Sigma, V, T) = V_T$.  \hspace{1cm} (4.21)

$M_2$ satisfies the following P.D.E.:

$$\left( \exp \left( - 2 \int_0^t r_u du \right) \right) \left( - 2r_t M_2 + \frac{\partial M_2}{\partial t} + \beta(\tilde{\alpha}_i^2 - \sigma_i^2) \frac{\partial M_2}{\partial \Sigma} + \frac{1}{2} \gamma^2 \sigma_i^2 \frac{\partial^2 M_2}{\partial \Sigma^2} + \sigma_i^2 \frac{\partial M_2}{\partial V} + \langle M_2, \Pi X \rangle \right) = 0,$$

with terminal condition $M_2(X, \Sigma, V, T) = V_T$.  \hspace{1cm} (4.22)

$M_3$ satisfies the following P.D.E.:

$$\left( \exp \left( - 4 \int_0^t r_u du \right) \right) \left( - 4r_t M_3 + \frac{\partial M_3}{\partial t} + \beta(\tilde{\alpha}_i^2 - \sigma_i^2) \frac{\partial M_3}{\partial \Sigma} + \frac{1}{2} \gamma^2 \sigma_i^2 \frac{\partial^2 M_3}{\partial \Sigma^2} + \sigma_i^2 \frac{\partial M_3}{\partial V} + \langle M_3, \Pi X \rangle \right) = 0,$$

with terminal condition $M_3(X, \Sigma, V, T) = V_T^2$.  \hspace{1cm} (4.23)
Note that with $X = e_j$ ($j = 1, 2, \ldots, N$),

\begin{align*}
    r_t &= < r, X_t >= r_j , \\
    \tilde{\alpha}_t &= < \tilde{\alpha}, X_t >= \tilde{\alpha}_j . \quad (4.24)
\end{align*}

Let $M_{ij} := M_i(e_j, \Sigma, V, T)$, where $i = 1, 2, 3$ and $j = 1, 2, \ldots, N$. Then, $M_i = (M_{i1}, M_{i2}, \ldots, M_{iN})$. Hence, $M_1$ satisfies the following system of $N$ coupled P.D.E.s:

\begin{align*}
    -r_j M_{1j} + \frac{\partial M_{1j}}{\partial t} + \beta (\tilde{\alpha}_j^2 - \sigma_t^2) \frac{\partial M_{1j}}{\partial \Sigma} + \frac{1}{2} \gamma^2 \sigma_t^2 \frac{\partial^2 M_{1j}}{\partial \Sigma^2} + \sigma_t^2 \frac{\partial M_{1j}}{\partial V} \\
    + < M_1, \Pi e_j >= 0 , \quad (4.25)
\end{align*}

with terminal condition $M_1(e_j, \Sigma, V, T) = V_T$.

$M_2$ satisfies the following system of $N$ coupled P.D.E.s:

\begin{align*}
    -2r_j M_{2j} + \frac{\partial M_{2j}}{\partial t} + \beta (\tilde{\alpha}_j^2 - \sigma_t^2) \frac{\partial M_{2j}}{\partial \Sigma} + \frac{1}{2} \gamma^2 \sigma_t^2 \frac{\partial^2 M_{2j}}{\partial \Sigma^2} + \sigma_t^2 \frac{\partial M_{2j}}{\partial V} \\
    + < M_2, \Pi e_j >= 0 , \quad (4.26)
\end{align*}

with terminal condition $M_2(e_j, \Sigma, V, T) = V_T$.

$M_3$ satisfies the following system of $N$ coupled P.D.E.s:

\begin{align*}
    -4r_j M_{3j} + \frac{\partial M_{3j}}{\partial t} + \beta (\tilde{\alpha}_j^2 - \sigma_t^2) \frac{\partial M_{3j}}{\partial \Sigma} + \frac{1}{2} \gamma^2 \sigma_t^2 \frac{\partial^2 M_{3j}}{\partial \Sigma^2} + \sigma_t^2 \frac{\partial M_{3j}}{\partial V} \\
    + < M_3, \Pi e_j >= 0 , \quad (4.27)
\end{align*}

with terminal condition $M_3(e_j, \Sigma, V, T) = V_T^2$. 34
Once $M_1$, $M_2$ and $M_3$ are solved from the above systems of $N$ coupled P.D.E.s, we can use them to approximate the prices of the variance swap and the volatility swap.

§5. Monte Carlo Experiment

In this section, we shall perform a Monte Carlo Experiment for the prices of the variance swaps and the volatility swaps implied by the regime-switching Heston’s stochastic volatility model. We shall document economic consequences for the prices of the variance swaps and the volatility swaps of a regime-switching in Heston’s stochastic volatility model by comparing the prices with those obtained from Heston’s SV model without regime-switching. We shall compute the prices of the variance swaps and the volatility swaps with various delivery prices under both the regime-switching Heston’s SV model and Heston’s SV model without switching regimes by Monte Carlo simulation. For illustration, we suppose that the number of regimes $N = 2$ throughout this section. The first and second regimes, namely $X_t = 1$ and $X_t = 2$, can be interpreted as the “Good” and “Bad” economic states, respectively. We also assume that Heston’s SV model without switching regimes coincides with the first regime of the regime-switching Heston’s SV model. In this case, we can investigate economic consequences for the prices of the variance swaps and the volatility swaps
when we allow the possibility that the dynamics of Heston’s SV model switches over time to the one corresponding to the “Bad” economic states. We generate 10,000 simulation runs for computing each price. All computations were done by C++ codes with GSL functions.

We shall assume some specimen values for the parameters of regime-switching Heston’s SV model and the one without switching regimes. When the economy is good (bad), the interest rate is high (low). Let \( r_1 \) and \( r_2 \) denote the annual interest rates for the “Good” state and the “Bad” state, respectively. Then, we suppose that \( r_1 = 5\% \) and \( r_2 = 2\% \). The appreciation rate of the underlying risky asset is high (low) when the economy is good (bad). In each case, the appreciation rate should be higher than the corresponding interest rate. Hence, we suppose that the annual appreciation rate \( \mu_1 = 7\% \) for the “Good” state and the annual appreciation rate \( \mu_2 = 5\% \) for the “Bad” state. When the economy is good (bad), the underlying risky asset is less (more) volatile. Hence, we suppose that \( \alpha_1 = 0.12 \) and \( \alpha_2 = 0.24 \). The speed of mean reversion \( \beta = 0.2 \) and the volatility of volatility parameter \( \gamma = 0.08 \). We also assume that the correlation coefficient \( \rho \) is negative and is equal to \(-0.5\). The transition probabilities of the Markov chain are \( \pi_{11} = 0.5, \pi_{12} = 0.5, \pi_{21} = 0.5, \pi_{22} = 0.5 \). The notational amount of the variance swap or the volatility swap is £1 million. We suppose that the current economic state \( X_0 = 1 \) and that the current
volatility level $V_0 = 0.12$. The delivery prices of the variance swap and the volatility swap range from 80% to 125% of the current levels of the variance and the standard deviation of the underlying risky asset, respectively. The time-to-expiry of both the variance swap and the volatility swap is 1 year. Since the regime-switching Heston stochastic volatility model is a continuous-time model, we need to discretize it when we compute the prices of the variance swaps and volatility swaps by Monte Carlo simulation. We suppose that the number of steps for the discretization is 20. Table 1 displays the prices of the variance swaps for various delivery prices implied by the regime-switching Heston stochastic volatility model and its non-regime-switching counterpart.

**Table 1: Prices of Variance Swaps with and without Switching Regimes**

<table>
<thead>
<tr>
<th>Delivery Prices in %</th>
<th>Prices with Switching Regimes in £ million</th>
<th>Prices without Switching Regimes in £ million</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>1.14556</td>
<td>0.847634</td>
</tr>
<tr>
<td>85</td>
<td>1.1431</td>
<td>0.845326</td>
</tr>
<tr>
<td>90</td>
<td>1.14165</td>
<td>0.843224</td>
</tr>
<tr>
<td>95</td>
<td>1.13823</td>
<td>0.840911</td>
</tr>
<tr>
<td>100</td>
<td>1.13607</td>
<td>0.838839</td>
</tr>
<tr>
<td>105</td>
<td>1.13167</td>
<td>0.836526</td>
</tr>
<tr>
<td>110</td>
<td>1.12934</td>
<td>0.834374</td>
</tr>
<tr>
<td>115</td>
<td>1.12657</td>
<td>0.832061</td>
</tr>
<tr>
<td>120</td>
<td>1.12196</td>
<td>0.829986</td>
</tr>
<tr>
<td>125</td>
<td>1.12082</td>
<td>0.82767</td>
</tr>
</tbody>
</table>
Table 2 displays the prices of the volatility swaps for various delivery prices implied by the regime-switching Heston stochastic volatility model and its non-regime-switching counterpart.

<table>
<thead>
<tr>
<th>Delivery Prices</th>
<th>Prices with Switching Regimes</th>
<th>Prices without Switching Regimes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>in £ million</td>
<td>in £ million</td>
</tr>
<tr>
<td>80</td>
<td>0.632453</td>
<td>0.467974</td>
</tr>
<tr>
<td>85</td>
<td>0.624144</td>
<td>0.461601</td>
</tr>
<tr>
<td>90</td>
<td>0.616364</td>
<td>0.455246</td>
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<tr>
<td>95</td>
<td>0.607529</td>
<td>0.448786</td>
</tr>
<tr>
<td>100</td>
<td>0.599301</td>
<td>0.442436</td>
</tr>
<tr>
<td>105</td>
<td>0.589936</td>
<td>0.436079</td>
</tr>
<tr>
<td>110</td>
<td>0.581685</td>
<td>0.429701</td>
</tr>
<tr>
<td>115</td>
<td>0.573195</td>
<td>0.423343</td>
</tr>
<tr>
<td>120</td>
<td>0.563777</td>
<td>0.416992</td>
</tr>
<tr>
<td>125</td>
<td>0.55599</td>
<td>0.410641</td>
</tr>
</tbody>
</table>

From Table 1 and Table 2, we see that the prices of the variance swaps and the volatility swaps implied by the regime-switching Heston stochastic volatility model are significantly higher than the corresponding prices of the variance swaps and the volatility swaps, respectively, implied by the standard Heston stochastic volatility without switching regimes. This reveals that a higher risk premium is required to compensate for the risk from the structural change of the volatility dynamics to the one with higher long-term volatility level due to the possible transitions of the states.
of the economy to the “Bad” state. This illustrates the economic significance of incorporating the switching regimes in the volatility dynamics for pricing variance swaps and volatility swaps.

§ 6. Further Research

For further investigation, it is interesting to explore and develop some criteria to determine the number of states of the Markov chain in our framework which will incorporate important features of the volatility dynamics for different types of underlying financial instruments, such as commodities, currencies and fixed income securities. It is also interesting to explore the applications of our model to price various volatility derivative products, such as options on volatilities and VIX futures, which are a listed contract on the Chicago Board Options Exchange. It is also of practical interest to investigate the calibration and estimation techniques of our model to volatility index options. Empirical studies comparing the performance of models on volatility swaps are interesting topics to be investigated further.

Acknowledgment

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References


