Two-level binomial trees

Phelim Boyle, Tak Kuen Siu and Hailiang Yang

In recent years, risk management has become of great importance to financial institutions and their regulators. We have seen the emergence of value-at-risk as a powerful and simple tool for capturing some dimensions of the risk of a financial firm’s entire portfolio. VAR provides a single number that makes it easier to explain the risk exposure of the enterprise to senior management. Duffie & Pan (1997) provide a good overview of VAR and its implementation in the case of a portfolio of derivatives. There are also several books on VAR, notably those by Brockman (2001) [not in ref box] and Dowd (1998).

Much of the literature on VAR deals with technical questions of implementation and these questions are certainly important ones for practitioners. There has been less work on the rationale for using VAR and discussion of the consequences of using this metric, but a number of authors have shown clearly that the traditional VAR measure has serious shortcomings. The first authors to clearly articulate the theoretical problems with VAR were Artzner et al. (1997, 1999). They showed that the traditional VAR measure violated one of four desirable axioms they developed for a coherent risk measure. Both Vorst (2000) working in discrete time and Basak & Shapiro (2001) working in continuous time found that if managers act as if VAR is a binding constraint they will make some bizarre and sub-optimal decisions. Föllmer & Leukert (1999) have also studied the consequences of VAR constraints on portfolio choice. This makes the analysis of more consistent risk measures for derivatives an interesting problem.

The generalised ‘scenarios’ expectation (GSE) proposed by Artzner et al. (1997, 1999) has a more solid theoretical basis than VAR. The GSE is defined as the supremum of the expected loss of a portfolio over a set of probability measures or generalised ‘scenarios’. It can be considered as a generalisation of the margin system Span (Standard Portfolio Analysis of Risk) developed by the Chicago Mercantile Exchange. The GSE is the representation form of the coherent risk measure, as proved in Artzner et al. (1999). Some special cases and further use of GSE have been developed recently. One of the examples is a measure based on the expected value of the losses above some level. Such measures include expected shortfall (ES) or conditional tail expectation (CTE), sometimes called tail VAR. This measure is known by different names, as different authors discovered it independently. One of the first to propose this measure was Longin (1997) [2001 in ref box] who labelled it BVAR. If the distribution is continuous, the ES coincides with the CTE. The CTE is defined as the average loss of a portfolio in the worst 100α% cases, for some (small) probability level α.

In practical applications, risk measures such as VAR, GSE and ES are normally considered in the framework of a single-period horizon such as one day or 10 days. In this article, we use a simple binomial tree model to provide a framework for discussing some of the issues that arise when we consider a portfolio containing derivatives. Specifically, we use a binomial model to explore the risk profile of a call option. One of the key steps for evaluating the risk or losses of the call option is to calculate the prices of the option. To get a reasonable approximation to the market price under a binomial model, one natural way is to divide the time interval of interest into many tiny sub-intervals. However, in practice the time horizon of interest is usually fixed and the risk profile is evaluated at the end of this horizon, say, one day ahead or 10 days ahead.

In this article, we introduce a two-level binomial model to provide a more flexible framework for risk measurement. The advantage of the two-level binomial model is to provide practitioners with the flexibility of adjusting the number of sub-intervals and keeping the time horizon of risk measurement fixed at the same time. This could be particularly useful when we deal with a portfolio with unhedged positions. Of course, significant external events that affect a company’s risk exposure can occur within these time periods. In addition, a firm can alter the composition of its portfolio inside these time windows. For a rigorous analysis of the issues involved in the multi-period case, we refer to Wang (1999) and Artzner et al. (2001).

We use the binomial model to examine the risk profile of derivatives. Using the two-level binomial tree model, the interaction among the three important elements of total risk management (Lo, 1999), namely price, probability and preference, becomes more transparent. We develop formulas for the different measures in this context and use simple numerical examples to illustrate the properties of different risk measures. The numerical results also provide an intuitive way to illustrate that ES is a particular case of GSE by choosing an appropriate set of generalised ‘scenarios’.

Real world versus risk-neutral world probabilities

Holton (1997) gives a contemporary definition of risk, defined as exposure to uncertainty. Besides distinguishing the concepts of risk and uncertainty, Holton’s notion of risk is intuitive and makes the mathematical formulation of risk more precise. In particular, risk can be formulated as a random variable representing the exposure and a probability space representing uncertainty. Another classical distinction between risk and uncertainty is due to Frank Knight [ref needed]. Knight proposed that the distinction between risk and uncertainty depends on whether the underlying probability is known or not. If the underlying probability is known, we regard the future contingencies as risk that is usually termed as ‘ontological’ uncertainty. Otherwise, it is considered as uncertainty. In our case, assume that the set of underlying probabilities are given in advance. Hence, it is more appropriate to adopt Holton’s notion of risk.

The probability apparatus provides an indispensable mathematical tool to describe or model uncertainty. Hence, it is one of the basic elements for a quantitative description of risk. To provide a clear conceptual foundation for risk measurement and management, one should clarify the meaning or interpretation of the underlying probability used for the calculation. Although there are some model-free or probability-free approaches to describe or measure risk in the finance literature (Artzner et al. 1997, 1999), the probability-based methods play a vital role in financial risk management. The nature of modern finance models leads naturally to different probability measures that play important but distinct roles in modern risk management. We now provide a brief discussion of these different probability measures and mention their role in risk management.

The risk-neutral probability measure plays an important role in pricing derivatives using the no-arbitrage principle. This is because when there is no arbitrage the current market price of a derivative can be written in terms of the discounted expected value of its payout at maturity where the expectation is taken over the risk-neutral measure. The relationship between
the risk-neutral probability and the price of the option becomes more direct when the market is complete in which there is a unique risk-neutral probability. For instance, the Black-Scholes-Merton formula for the price of a call option can be derived by finding the expected value of the option payout under this risk-neutral measure. This risk-neutral probability measure arises directly from market prices and it is not surprising that it is useful in reproducing market prices. The risk-neutral measure is often called the \( Q \) measure and it is the one implicitly used in pricing models. The prices quoted by traders in the front office are based on the \( Q \)-measure.

To measure the risk of an option, we move from the risk-neutral world to the real world where, in contrast to the risk-neutral world, there is no natural choice for the assignment of probability measures (Lo, 1999). Different risk measures and their associated interpretations are obtained if different probabilities are used for the calculations. Hence, one should pay attention to the distinction between the subjective and objective interpretations of probabilities when one assigns probability measures for risk measurement, especially when we consider the GSE (see below), where a family of probabilities are used for the calculation. Besides different interpretations of probabilities, the information structure is also important for the calculation our risk measures. Here, for the purpose of pricing options, we only require the weak form of market efficiency and, hence, the information structure is generated by the history of the underlying stock prices. Lo (1999) pointed out that the risk management literature often focuses on the algebra of probabilities, the calculus of probability distributions and their statistical estimations. The distinction between the real world probability and the subjective probabilities is often neglected.

Here, the real world probability \( P \), which is also called the objective/statistical probability, is the underlying probability law that governs the realisation of the stock price movement. It is related to some statistical phenomena that are repeatable and, hence, we can estimate it by using some statistical methods based on the known information or past data that is usually unknown to an investor in real situations. In contrast to the real world probability, the assignment of subjective probabilities need not be subject to a common agreement. It seems that the choice of a subjective probability depends on an agent’s risk preference and subjective views. However, in our case, where the no-arbitrage principle is needed for pricing options, the choice of subjective probabilities is not completely arbitrary. We require that a subjective probability should satisfy the following basic property of probability:

\[
P(A) + P(A^c) = 1
\]

where \( A^c \) is the complement of an event \( A \).

This is a consequence of the so-called Dutch book theorem, which requires that subjective probabilities should satisfy the above basic property of probability in order to avoid arbitrage opportunities (Lo, 1999).

### Risk measures

Here, we discuss VAR and another risk measure based on the expected value of losses above some threshold that is superior to VAR. This last measure is known by different names, such as tail VAR, BVAR, CVAR and CTE. In the case when the distribution of losses is discrete such as in our binomial case, we need to adjust this measure to allow for the possibility that there may be a discrete mass of probability at the critical threshold. For the computations of VAR and tail VAR, only one reference probability is used. In practice, one usually considers the real world probability \( P \) as the reference probability. Here, we also use the \( P \) measure for calculating both VAR and tail VAR.

VAR is basically a quantile measure and its shortcomings have been well documented. Assume that \( X \) is a random variable that denotes the future value of the profit at the end of some fixed time horizon. Suppose that \( \alpha \equiv A\% \in (0, 1) \). The VAR of the portfolio is given by:

\[
\text{VAR}^{(\alpha)}(X) = -\chi^{(\alpha)}(X)
\]

where:

\[
\chi^{(\alpha)}(X) = \sup \{ x | P[X \leq x] \leq \alpha \}
\]

Note that \( P \) is the real world probability.

The CTE or ES proposed by Artzner et al (1997, 1999) and Wirch & Hardy (1999) is a more satisfactory risk measure. It represents the expected value of the losses given that the losses are below some quantile. For continuous distributions, it is defined as follows:

\[
\text{CTE}^{(\alpha)}(X) = -E_P \{ X | X \leq \chi^{(\alpha)}(X) \}
\]

In the case of discrete distributions, the probability of the event \( \{ X \leq \chi^{(\alpha)}(X) \} \) may be larger than \( \alpha \%, \) in which case the definition of ES needs to be amended. The amended measure, proposed by Wirch & Hardy (1999), consists of a weighted combination of (2) and (3):

\[
- E_P \{ X | X \leq \chi^{(\alpha)}(X) \} \lambda_u - (\lambda_u - 1) \text{VAR}^{(\alpha)}(X)
\]

where:

\[
\lambda_u = \frac{P[X \leq \chi^{(\alpha)}(X)]}{\alpha}
\]

Note that for a continuous distribution \( \lambda_u = 1 \) and we revert to equation (2). The weighted average of VAR and BVAR is necessary to make the ES a coherent risk measure when the corresponding loss distribution is discrete. We will use the amended version of the ES to evaluate the risk of a call option and provide the corresponding numerical results later. Different versions of the proof of the coherence of the ES can be found in Artzner et al (1999), Delbaen (1999) and Acerbi & Tasche (2001).

Acerbi & Tasche (2001) also suggested the same modification and described it by the term ES. The Acerbi-Tasche definition of ES is:

\[
\rho_{\alpha}^{\text{ES}} = - \frac{1}{\alpha} \left[ E_P \left[ X \mathbf{1}_{[X \leq \chi^{(\alpha)}(X)]} \right] - \chi^{(\alpha)}(X) \left( P[X \leq \chi^{(\alpha)}(X)] - \alpha \right) \right]
\]

where \( \mathbf{1}_{[X \leq \chi^{(\alpha)}(X)]} \) is the indicator function of the event \( \{ X \leq \chi^{(\alpha)}(X) \} \).

Rockafeller & Uryasev (2001) also proposed the same modification to account for discrete distributions and used the term \( \alpha \)-conditional value-at-risk. One can show that the expressions derived by Wirch & Hardy for
the CTE when the distribution is discrete is equivalent to the ES as defined by Acerbi & Tasche (2001), and both, in turn, are equivalent to the \( \alpha \)-conditional value-at-risk proposed by Rockafeller & Uryasev.

Two-level binomial tree model

The origins of the idea of using a discrete-time binomial tree and the contribution of William Sharpe are described by Rubinstein (2001). Cox, Ross & Rubinstein (1979) wrote an important article on the binomial model and showed how it provided a simple and powerful valuation tool that converges in the limit as the number of steps becomes large to the Black-Scholes-Merton model. Hence the binomial tree model is often referred to as the Cox-Ross-Rubinstein model. We propose a two-level binomial tree for analysing the risks of derivatives inside the VAR horizon. We assume two assets: the risk-free bond \( B \) and a risky stock \( S \).

We denote \( T \) as the time index set \( \{0, 1, 2, \ldots, T\} \) in the first level of the binomial model. For each \( k \in T \setminus \{T\} \) as the time horizon for risk measurement. Later, we will subdivide this interval into smaller periods. Over the time horizon \( [k, k+1] \), we suppose that positions will not be liquidated or adjusted. In the second level of the binomial model, we divide each time interval \( [k, k+1] \) into \( m \) sub-intervals of equal length \( h \), where \( mh = 1 \). For notational simplicity, we omit the subscript \( h \) throughout this article and let \( T_m \) be the time index set \( \{0, 1, 2, \ldots, km, km+1, \ldots, (k+1)m, \ldots, Tm\} \) in the second level of the binomial model. Thus, in the second level of the binomial model the basic time unit is of length \( h \) in terms of the units used in the first level.

We now construct the binomial tree model based on the finer partition. Let \( r \) be the constant rate of return of the bond \( B \) over each sub-interval. For each sub-interval \( [n, n+1] \) in the second level of the binomial model, the bond price process \( \{B_n\}_{n \in T_n} \) and the stock price process \( \{S_n\}_{n \in T_n} \) satisfy:

\[
B_{n+1} = B_n e^{r \Delta t} \\
S_{n+1} = \begin{cases} S_n e^{d \Delta t} & \text{if the stock goes up at time } n+1 \\ S_n e^{-d \Delta t} & \text{if the stock goes down at time } n+1 \end{cases}
\]

where \( \hat{d} = 1 + r + d < 1 \) and \( d < \hat{d} < u \). The condition \( d < \hat{d} < u \) is equivalent to the absence of arbitrage opportunities. Figure 1 shows an example of a two-level binomial tree with \( T = 3 \) and \( m = 2 \). The two-level binomial model provides the flexibility of adjusting \( m \) according to the desired degree of accuracy in the approximation to the continuous-time model. It also provides a flexible framework for controlling the precision in the estimation of the risk measures under the uncertainty of the real world probability, which is usually the case in practical situations. The estimation result is desirable when a significant number of sub-intervals are used in the two-level binomial model.

Expected shortfall

We now use the simple binomial tree model shown in figure 1 to consider the risk of an unhedged position. In practice, banks usually hedge, due to the transaction and operational costs, they may not always adjust their positions during the risk measurement horizon. Thus the measurement of an unhedged position is important. For illustration, we focus on a very simple portfolio. We consider a single European-style call option writ-

### 1. A two-level binomial tree with \( T = 3 \) and \( m = 2 \)

\[
C_{km} = \frac{1}{Tm-km} \sum_{j=0}^{Tm-km} \left( C_{km} - \frac{Tm-km}{j} \left[ (1-q) \left( \frac{S_{km} e^{d \Delta t-j}}{C_{km}} - K \right) \right] \right)
\]

where \( q = \hat{d} - d(u - d) \) is the probability of an upward movement of the stock \( S \) at the end of each sub-interval under the \( Q \)-measure.

Under the \( P \)-measure, the conditional distribution of \( \Delta C_{km} \) given \( F_{km} \) is:

\[
\Delta C_{km} = C_{km} - r^{-\alpha} \Delta C_{(k+1)m} \left( S_{km} e^{d \Delta t-j} \right)
\]

with probability \( \left\{ m \right\} \left( 1 - \left( 1 - p \right)^{-j} \right) \), where \( p \) represents the probability of an upward movement of the stock at the end of each sub-interval under the \( P \)-measure.

We are now ready to give explicit expressions for the various risk measures in terms of this binomial model. First, we define the VAR measure for the position \( \Delta C_{km} \) with probability level \( \alpha \) and with respect to the real world probability \( P \):

\[
\text{VAR}_{\alpha, P} (\Delta C_{km} | F_{km}) = \text{sup} \left\{ x | P (\Delta C_{km} > x | F_{km}) > \alpha \right\}
\]

The expression for the ES for the unhedged position of the call option \( C \) over the time horizon \( [km, (k+1)m] \) is:

\[
\text{ES} (\Delta C_{km} | F_{km}) = \alpha^{-1} \left( [ \Gamma (\Delta C_{km} | F_{km}) > \text{VAR}_{\alpha, P} (\Delta C_{km} | F_{km}) ] | F_{km} \right) + \text{VAR}_{\alpha, P} (\Delta C_{km} | F_{km}) (\alpha - P (\Delta C_{km} > \text{VAR}_{\alpha, P} (\Delta C_{km} | F_{km}) ) \right)
\]

where \( I(A) \) is the indicator of an event \( A \) and the expectation is taken with respect to the real world probability \( P \).

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\]

where \( I(A) \) is the indicator of an event \( A \) and the expectation is taken with respect to the real world probability \( P \).

The risk manager can get more complete details of the riskiness of the portfolio by investigating the conditional distribution of \( \Delta C_{km} \) given \( F_{km} \). However, it is cumbersome to do so if \( m \) is large, and so a risk measure is a useful summary measure.
A. Comparison of expected shortfall and VAR for various values of \( p \) and \( \alpha \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \alpha )</th>
<th>( 0.01 )</th>
<th>( 0.05 )</th>
<th>( 0.10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>2.795653 (2.795653)</td>
<td>2.795653 (2.795653)</td>
<td>2.795653 (2.795653)</td>
<td>2.795653 (2.795653)</td>
</tr>
<tr>
<td>0.4</td>
<td>2.795653 (2.795653)</td>
<td>2.795653 (2.795653)</td>
<td>2.616325 (1.9989324)</td>
<td>2.616325 (1.9989324)</td>
</tr>
<tr>
<td>0.5</td>
<td>2.795653 (2.795653)</td>
<td>2.492380 (1.9989324)</td>
<td>2.492380 (1.9989324)</td>
<td>2.492380 (1.9989324)</td>
</tr>
<tr>
<td>0.6</td>
<td>2.795653 (2.795653)</td>
<td>2.154460 (1.9989324)</td>
<td>2.154460 (1.9989324)</td>
<td>2.154460 (1.9989324)</td>
</tr>
<tr>
<td>0.7</td>
<td>2.185262 (1.9989324)</td>
<td>1.591582 (0.852669)</td>
<td>1.591582 (0.852669)</td>
<td>1.222126 (0.852669)</td>
</tr>
<tr>
<td>0.8</td>
<td>1.591582 (0.852669)</td>
<td>1.222126 (0.852669)</td>
<td>1.222126 (0.852669)</td>
<td>1.222126 (0.852669)</td>
</tr>
</tbody>
</table>

B. Values of GSE and ES for different parameter values

<table>
<thead>
<tr>
<th>( \Lambda )</th>
<th>([p^\alpha, p^\beta])</th>
<th>GSE</th>
<th>ES (( \alpha ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>[20, 25]</td>
<td>[2.98 x 10^{-8}, 9.54 x 10^{-7}]</td>
<td>2.795653</td>
<td>2.795653 (0.01)</td>
</tr>
<tr>
<td>[3, 3.816]</td>
<td>[0.071002, 0.125]</td>
<td>2.492384</td>
<td>2.492380 (0.05)</td>
</tr>
<tr>
<td>[2, 3]</td>
<td>[0.125, 0.25]</td>
<td>2.242985</td>
<td>2.241302 (0.10)</td>
</tr>
</tbody>
</table>

Let \( j_a \) be given by:

\[
j_a = \max \left\{ j \in J \left\{ \sum_{j=1}^{m} f_{11} \right\} \geq \text{VAR}_{p, \alpha} \left\{ \Delta C_{i,m} \right\} \right\}
\]

where \( J \) represents the set \( \{0, 1, 2, \ldots, m\} \). Then, we obtain the following explicit form for (7):

\[
\rho_{\text{ES}} \left( \Delta C_{i,m}, F_{i,m} \right) = \frac{1}{\sum_{j=1}^{m} f_{11}} \left[ \frac{\sum_{j=1}^{m} f_{11} \left( 1 - p \right)^{j-1} \left( 1 - p \right)^{m-j} \left( \sum_{i=0}^{m} \left( T_m - (k+1)m \right) \right) q_i \left( 1 - q \right)^{\left( T_m - (k+1)m \right) - K} \right] + \Delta C_{i,m} \left( u^d, d^m \right) \right]
\]

We use a simple numerical example to illustrate the calculation of the ES for this case. We also compare the numerical values of ES with those of VAR for different values of the parameters \( p \) and \( \alpha \).

Example 1. Consider a European-style call option that will mature in two months. The initial stock price \( S \) is 25, the strike price is 22 and \( \alpha \) is positive (zero, negative) if the agent is risk-averse (risk-neutral, risk-seeking). Let \( \lambda \) be the (log) \( q \)-log \( p \) or, equivalently, \( \rho_{\lambda} \). As \( \lambda \) is the set \( \{ \lambda \} \) containing only one element \( \lambda \), the GSE (10) becomes zero since \( E_{\lambda} (\Delta C_{i,m}, F_{i,m}) = 0 \). Thus, the risk-neutral probability can serve as a reference point for the GSE. Suppose an agent chooses an index set \( \Lambda \) for implementing the GSE. Then, the GSE associated with the index set \( \Lambda \) is positive (zero, negative) if the agent is risk-averse (risk-neutral, risk-seeking). This property plays a similar role to the utility functions in financial economics.

By calculating the expectation with respect to each subjective probability \( P_{\lambda} \) and noticing that the supremum is attained when \( \Lambda \) equals \( \lambda \), we can show that the GSE (10) can be expressed in the following form:

\[
\rho_{\lambda} \left( \Delta C_{i,m}, F_{i,m} \right) = \frac{1}{\sum_{j=1}^{m} f_{11}} \left[ \sum_{j=1}^{m} f_{11} \left( 1 - p \right)^{j-1} \left( 1 - p \right)^{m-j} \left( \sum_{i=0}^{m} \left( T_m - (k+1)m \right) \right) q_i \left( 1 - q \right)^{\left( T_m - (k+1)m \right) - K} \right] + \Delta C_{i,m} \left( u^d, d^m \right) \right]
\]

We now develop a simple numerical example to illustrate the implementation of the GSE (11) and how to relate the GSE to the ES with a
Example 2. Using the same parameters and the same two-level binomial model as before, we can compare the numerical values of ES and GSE for different parameters α and index sets Λ. For illustration, we let the real world probability p be 0.5. In Table 2, we provide the set of numerical values of the GSE.

From (10), the calculation of GSE involves the upper limit b rather than the lower limit α of the interval [α, b]. A small value for the probability level α corresponds to risk-averse behaviour and hence a large ES. To make the value of the GSE close to that of the ES, we need to make the GSE more conservative by enlarging the set [α, b] of ‘scenarios’ or, equivalently, choosing a large value for b. When the upper limit b becomes larger, the distorted probability pb becomes smaller. This makes the probability of incurring large losses of the call position increase.

Risk measures for other vanilla European-style derivatives, such as put options, binary options and vanilla options, can be defined and calculated in a similar fashion. Both the ES and the GSE are defined from the viewpoint of a buyer. From the writer’s point of view, the discounted net loss becomes smaller. This means that the discounted net loss from the writer’s viewpoint is the negative of the discounted net loss from the buyer’s perspective. The corresponding risk measure is calculated based on the discounted net loss ΔC Leon.

Conclusions
Risk measurement and management are important topics in modern financial practice. Trading derivatives is a risky business due to its high leverage effect. There are significant risks associated with holding a portfolio of unhedged derivatives. The analysis of the risk profiles of such portfolios is an important practical issue. We have introduced a two-level binomial model that is suitable for measuring the risk exposure of unhedged positions of derivatives. Within our discrete-time binomial framework, the interaction among the three p’s of total risk management becomes clearer. We determined the price of an option using the binomial tree valuation formula while we generated a family of subjective probabilities from the real world probability according to an agent’s risk preference and subjective views using the distortion approach. We also discussed the coherent risk measures that are theoretically superior to VAR in order to ease the shortcomings of VAR when the risk profile contains derivatives. The ES and the GSE (as pointed out in Artzner et al., 1999), ES can be considered as a special case of GSE are discussed.

The two-level binomial tree provides a simple but powerful way to illustrate the calculation of these risk measures for a single derivative position. Risk practitioners can apply the same principle to calculate these risk measures for other derivatives positions. Naturally, the situation is much more complicated in the case of a large portfolio with several derivatives positions. If the portfolio contains American-style derivatives, we should take into account the possibility of early exercise in both the pricing part and risk-measurement part. In this case, the two-level binomial tree is useful for incorporating the possibility of early exercise at intermediate dates.

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