On Pricing Derivatives under GARCH Models: A Dynamic Gerber-Shiu’s Approach

Tak Kuen Siu∗, Howell Tong† and Hailiang Yang‡

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Purpose of the paper: This paper proposes a method for pricing derivatives under the GARCH assumption for underlying assets in the context of a “dynamic” version of Gerber-Shiu’s option-pricing model.

Key words: Conditional Esscher transforms, Option pricing, GARCH Models, Infinitely divisible distributions, Dynamic utility framework

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∗Tak Kuen Siu, Ph.D., is a Research Fellow in the Liu Bie Ju Centre for Mathematical Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong, e-mail: tksiu@cityu.edu.hk
†Howell Tong, Hons. F.I.A., Ph.D., is the Chair Professor of the Department of Statistics and Actuarial Science, The University of Hong Kong, Pokfulam Road, Hong Kong, e-mail: htong@hku.hk. He is also a Chair Professor of the Department of Statistics, London School of Economics, U.K.
‡Hailiang Yang, A.S.A., Ph.D., is an Associate Professor in the Department of Statistics and Actuarial Science, The University of Hong Kong, Pokfulam Road, Hong Kong, e-mail: hlyang@hkusua.hku.hk.
This paper proposes a method for pricing derivatives under the GARCH assumption for underlying assets in the context of a “dynamic” version of Gerber-Shiu’s option-pricing model. Instead of adopting the notion of local risk-neutral valuation relationship (LRNVR) introduced by Duan (1995), we employ the concept of conditional Esscher transforms introduced by Bühlmann et al. (1996) to identify a martingale measure under the incomplete market setting. One advantage of our model is that it provides an unified and convenient approach to deal with different parametric models for the innovation of the GARCH stock-price process. Under the conditional normality assumption for the stock innovation, our pricing result is consistent with that of Duan (1995). In line with the celebrated Gerber-Shiu’s option pricing model, we can justify our pricing result within the dynamic framework of utility maximization problems which makes the economic intuition of our pricing result more appealing. In fact, the use of the Esscher Transformation for option valuation can also be justified by the minimization of the relative entropy between the statistical probability and the risk-neutral pricing probability. Numerical results for the comparision of our model with the Black-Scholes option pricing model are presented.

Key words: Conditional Esscher transforms, Option pricing, GARCH Models, Infinitely divisible distributions, Dynamic utility framework
§1. Introduction

Option pricing is one of the major areas in modern financial theory and practice. Since Fischer Black, Myron Scholes and Robert Merton introduced their path-breaking work on option-pricing model in 1973, there is an explosive growth in the trading activities on derivatives in the worldwide financial markets. The main contribution of the seminal work of Black and Scholes (1973) and Merton (1973) is the introduction of a preference-free option-pricing formula which does not involve an investor’s risk preferences and subjective views. Due to its compact form and computational simplicity, the Black-Scholes formula enjoys a great popularity in the finance industries. One of the important economic insight underlying the preference-free option-pricing result is the concept of perfect replication of contingent claims by continuously adjusting a self-financing portfolio under the no-arbitrage principle. Cox et al. (1979) provided further insights in the concept of perfect replication by introducing the notion of risk-neutral valuation and establishing its relationship with the no-arbitrage principle in a transparent way under a discrete-time binomial setting. Harrison and Kreps (1979) and Harrison and Pliska (1981) established a solid mathematical foundation for the relationship between the no-arbitrage principle and the notion of risk-neutral valuation using the modern language of probability theory. They proposed the “Fundamental theorem for asset pricing” which states that the absence of arbitrage opportunities is equivalent to the existence of an equivalent martingale measure. If the securities market is complete, there is a unique martingale measure and hence the unique price of any contingent claim is given by its expected discounted payoff at expiry under the martingale measure. However, the assumption of market completeness is questionable in the real-world securities market. Under an incomplete market, there is more than one equivalent martingale measure and hence a range of no-arbitrage prices for a contingent claim. One crucial issue is to identify an equivalent martingale measure which gives an economically consistent and justifiable price for the contingent claim.

Föllmer and Sondermann (1986), Föllmer and Schweizer (1991) and Schweizer (1996) identified a unique equivalent martingale measure by minimizing the variance of the hedging loss. In fact, the quadratic loss of the hedge position can be related to the concept
of a quadratic utility (see Boyle and Wang (2001)). An interesting paper by Gerber and Shiu (1994) provided an elegant way to choose an equivalent martingale measure using the Esscher transformation, a time-honour tool in actuarial science introduced by Esscher (1932), in an incomplete market setting. Their approach provides market practitioners with the flexibility of choosing a wide variety of parametric models within the class of infinitely divisible distributions. It also provides practitioners with a very convenient tool for option valuation. The novelty of their approach is that the option price chosen by the Esscher transformation can be justified by maximizing the expected power utility of an economic agent. More importantly, their work provides an important insights in bridging the gap between the financial and insurance pricing problems in an incomplete market. Bühlmann et al. (1996) generalized the classical notion of Esscher transform to stochastic processes. They introduced the concept of conditional Esscher transforms in order to incorporate a richer theory of semimartingale under the no-arbitrage condition in the context of Gerber-Shiu’s option-pricing model. Bühlmann et al. (1998) investigated the use of Esscher transformation in discrete finance models and established a solid foundation of its use based on some economic arguments. Shiryaev (1999) and Jacod and Shiryaev (2003) provided a rigorous theoretical background for the use of Esscher transformation for choosing martingale pricing measure via the modern language of probability theory.

Market incompleteness can arise through a variety of extensions to the standard Black-Scholes economy. One major stream of the extensions is the relaxation of the Black-Scholes assumption of Geometric Brownian Motion (GBM). Numerous models have been proposed to replace the stringent GBM assumption in the finance and actuarial science literature. In particular, the changing volatility (or heteroskedastic) models and the Lévy-type models (or its extension, the class of infinitely divisible distributions) are two major classes of general models which are of great interests from both theoretical and practical viewpoints. According to Cox (1981), the changing volatility models can be classified into the observation-driven ARCH-type models of Engle (1982) and its variants

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1Siu et al. (2001) provided another direction of generalizing the classical notion of Esscher transform, namely random Esscher transform, by assuming the Esscher parameter as a random variable. One application of random Esscher transform is to generate a family of random generalized “scenarios” for risk measurement.
and generalizations, for instances, GARCH models of Bollerslev (1986) and Taylor (1986), and parameter-driven stochastic volatility (SV) models proposed by Taylor (1982, 1986). Under SV models, the logarithmic returns of a financial asset are assumed to be the product of two independent innovations processes. In practice, the most popular SV model is the lognormal SV model or the autoregressive random variance model introduced by Taylor (1982, 1986). Recently, the ARCH-type models has gained its empirical success in modelling many “stylized” facts in financial time series, especially in modelling the changing variance and covariance structure of financial time series. It is a more realistic model for modelling the dynamics of underlying assets compared with GBM. However, the discrete-time and continuous-state nature of the ARCH-type (GARCH) models makes the market incomplete and hence complicates the pricing issue.

Duan (1995) was the first to provide a solid theoretical foundation for option valuation in the context of GARCH models. The seminal work by Duan (1995) generalized the concept of risk-neutral valuation and introduced the notion of locally risk-neutral valuation relationship (LRNVR) which provides a sound economic argument to choose a particular equivalent martingale measure under the GARCH model with conditionally normal stock innovation. The martingale measure $Q$ with the LRNVR in Duan (1995) satisfies the following four conditions:

1. $Q$ is equivalent to the statistical probability measure $P$ on the sample space $\Omega$.
2. $\frac{S_t}{S_{t-1}}$ is lognormally distributed under $Q$.
3. $E_Q(\frac{S_t}{S_{t-1}}|\Phi_{t-1}) = e^{r}$, almost surely.
4. $Var_Q[ln(\frac{S_t}{S_{t-1}})|\Phi_{t-1}] = Var_P[ln(\frac{S_t}{S_{t-1}})|\Phi_{t-1}]$, almost surely; that is, the conditional variance of the logarithmic return $ln(\frac{S_t}{S_{t-1}})$ is invariant under the change of probability measures from $P$ to $Q$ almost surely.

Under the assumptions that the representative agent is an expected utility maximizer, that the utility function satisfies certain preference structure and that the logarithmic aggregate consumption are normally distributed with constant mean and variance, Duan
(1995) provided a rigorous theoretical foundation and economic justification of the validity of LRNVR. For details, see Theorem 2.1 of Duan (1995). In the Duan’s GARCH option pricing model, the conditional variance process of the underlying asset under the risk-neutralized pricing probability specified by LRNVR is a non-linear asymmetric GARCH model with noncentral chi-square innovation and the parameter of noncentrality being the unit risk premium for the underlying asset while the conditional variance process under the original statistical probability is a linear GARCH model with chi-square innovation. See Theorem 2.2 in Duan (1995) for the conditional variance process of the underlying asset under the risk-neutralized pricing probability measure $Q$. It has also been proved in Duan (1995) that the discounted price process for the underlying asset is a martingale under $Q$.

One feature of the LRNVR is that it preserves the one-period-ahead (local) conditional variance of the stock-price dynamic under changing the statistical probability measure to the risk-neutralized pricing measure. Since the global conditional variance process depends on the constant unit risk premium of the underlying asset, the GARCH option price is not preference-free. However, one can estimate the unit risk premium directly from the empirical data since it is part of the conditional expected return on the underlying asset. In fact, the relevance of unit risk premium of the underlying asset in the pricing of derivatives under GARCH model is not simply due to market incompleteness. Kallsen and Taqqu (1998) proposed a continuous-time version of the GARCH model and showed that the continuous-time version of the GARCH model constitutes a complete market. They also obtained the arbitrage-free option price under the completeness of the market. The no-arbitrage price obtained by Kallsen and Taqqu (1998) is in complete agreement with the GARCH option price in Duan (1995) and depends on the unit risk premium of the underlying asset. Duan (2001) constructed a semi-recombining binomial lattice to complete the market and showed that the no-arbitrage prices of any contingent claims under the semi-recombining binomial lattice depend on the unit risk premium of the underlying asset; that is, the risk-neutral valuation is not an inherent feature of market completeness. It is interesting to note that the limiting model of the semi-recombining binomial lattice in Duan (2001) coincides with the continuous-time version of the GARCH

Another feature of the GARCH option price is that it is non-Markovian in nature. In particular, the GARCH option price depends on the information set generated by the past and current prices of the underlying asset and its past, current and one-step-ahead values of the conditional variance process up to the orders of the GARCH model \(^2\) (see Corollary 2.3 of Duan (1995)). Finally, it is worth pointing out that the inclusion of the unit risk premium in the GARCH option price can indeed explain empirically some “stylized” systematic pricing biases of the Black-Scholes model, for instance, underpricing of options with short expiry, out-of-money options and options written on the underlying asset with low volatility and the U-shape behavior of implied volatility with respect to strike price (see Duan (1995)).

In this paper, we propose an alternative approach to the valuation of derivatives under a general class of GARCH models in the context of a “dynamic” version of Gerber-Shiu’s option-pricing model. We adopt the concept of conditional Esscher transforms introduced by Bühlmann et al. (1996) to identify an equivalent martingale measure and hence the price for a contingent claim or any security under the incomplete market. Here, the market incompleteness is induced by the discrete-time and continuous-state nature of the GARCH model. The advantage of the use of conditional Esscher transforms for picking an equivalent martingale measure is its capability in incorporating different infinitely divisible distributions for the GARCH innovations in an unified and convenient manner. It has been mentioned in Kallsen and Shiryaev (2002) that Esscher transformations enjoy a desirable mathematical property that it can be computed easily even for the class of general semimartingales since the whole density process is given in a form that enables the use of Girsanov’s theorem in a convenient manner. In fact, one only needs to solve an equation for the Esscher parameter that ensures the martingale property of the securities price processes. Duan (1999) provided a very flexible option-pricing model for the generalization of the standard GARCH option-pricing model in Duan (1995) in order to incorporate the class of GARCH models with conditional fat-tailed innovation. Here, we focus on the

\(^2\)For instances, the GARCH option price under the GARCH(1, 1) assumption for the price process of the underlying asset depend on the current price of the asset and one-step-ahead conditional variance
class of GARCH processes with conditionally infinitely divisible distributed innovation, for instances, the conditional normal innovation and the shifted gamma innovation. The GARCH model with shifted gamma innovations can be used to capture the skewed behavior in the stock innovation. For the case of conditional normality for the GARCH innovation, we start with the same GARCH process as in Duan (1995) under the statistical probability measure. The pricing result in this case is in complete agreement with that of Duan (1995). The concept of conditional Esscher transforms can also be employed to obtain the same risk-neutralized GARCH process for the price of the underlying asset as that in Heston and Nandi (2000). We can justify our choice of the equilibrium price by conditional Esscher transform that maximizes the conditional one-period expected power utility of an economic agent recursively. The concept of conditional Esscher transforms can indeed provide an interesting linkage between the changing conditional variances of the GARCH model and the sequence of the conditional risk-averse parameters in the dynamic power utility framework. This constitutes an intuitive economic argument to justify our choice of the equilibrium price.

We do not claim here that our economic argument for the justification of our GARCH option price is perfect. In fact, the use of the Esscher Transformation for option valuation can also be justified by the maximization of relative entropy between the statistical probability and the risk-neutral pricing probability. See Chan (1999) for detailed discussion on the relationship between the use of Esscher Transformation for option valuation and the minimization of the relative entropy. It can also be shown that the structure of the paths of the Lévy-type stock price models is not changed much under the Esscher change of measure (see Kallsen and Shiryaev (2002)). In our model, the parametric forms of the GARCH innovation are preserved under the Esscher change of measures. In Stutzer (1995), the concept of Esscher transformation has been employed for a variational characterization of stochastic discount factors and their closely related state price densities in asset pricing models based on the minimization of the Kullback-Leibler Information Criterion. In a recent paper by Cherny and Maslov (2003), the use of Esscher transformation has been justified in a general discrete-time asset pricing model with multiple risky assets via the concept of the minimal entropy martingale measure and the problem of the
exponential utility maximization. They pointed out that the use of conditional Esscher transform for the construction of martingale pricing measure is similar to the method of constructing the minimal entropy martingale measure. It has also been mentioned in Cherny and Maslov (2003) that the problems of finding the minimal entropy martingale measure and the exponential utility maximization are dual to each other. We aim to highlight some implications to the interplay between mathematical finance and actuarial science by exploring the relationship between actuarial and financial pricing in an incomplete setting. See Embrechts (2000) for a detailed discussion about the interaction between financial and actuarial pricing. We organise this paper in the following way.

The next section presents the general setting of our model. We consider a discrete-time economy consisting of two primary assets, namely a risk-free bond and a risky stock in the context of Gerber-Shiu’s option-pricing model. We assume that the distribution of the stock innovation is infinitely divisible with a finite moment generating function. First, we will present the generic setting of our GARCH model. The construction of risk-neutralized probability measure using the concept of conditional Esscher transform and the martingale condition for the equilibrium price will be presented next. In particular, we focus on an arbitrary (possibly path-dependent) European option. Finally, we justify our equilibrium price by the dynamic utility framework. For the ease of exposition, we will keep the use of the terminologies in measure theory or probability theory minimal. Section three discusses some special cases of our model, namely the conditional normal and the conditional shifted gamma cases. In both cases, we can describe the dynamic of the logarithmic stock-price returns under the risk-neutralized probability measure that is useful for the simulation purpose. As in Gerber-Shiu’s option-pricing model, we can preserve the parametric forms of the distributions for the GARCH innovation under the change of probability measures from the statistical probability measure and the risk-neutralized probability measure. In section four, we present some numerical results for the comparison of our model with the standard Black-Scholes option pricing model. The final section concludes this paper and proposes some possible topics for further research.

§2. Option Pricing under GARCH Model via Conditional Esscher Transforms
We consider a discrete-time financial model consisting of one risk-free bond \( B \) and one risky stock \( S \) in the context of Gerber-Shiu’s option-pricing model. For the sake of generality, we do not impose any stringent parametric assumption on the innovations process of the underlying stock \( S \). We only assume that the distribution of the innovations process is infinitely divisible and that the moment generating function of the infinitely divisible distribution exists. The latter condition is a reasonable one since, otherwise, the conditional expectation of the one-period simple returns for the underlying asset is unbounded (see Duan (1999)). Bollerslev (1987), Baillie and Bollerslev (1989), Hsieh (1989), Baillie and DeGennaro (1990) and Wang et al. (1996) pointed out that the conditional normality assumption for the asset innovation fails to incorporate the leptokurtic behavior of asset returns. Another shortcoming of the conditional normality assumption is that it fails to incorporate the skewness in the conditional innovation of some financial time series, such as the exchange-rate time series (see Wang et al. (2000)). For the exchange-rate time series, it may be more appropriate to consider other conditional distributions for the innovation, such as conditional shifted gamma distributions, which can incorporate the positively skewed behavior of the exchange-rate time series. Duan (1999) extended the notion of LRNVR to generalized LRNVR (GLRNVR) in order to deal with the conditional fat-tailed distributions, which are continuous with mean zero and variance \( h_t \), of the GARCH innovation. The main idea of his approach is the use of functional transformations for a conditionally fat-tailed GARCH innovation in order to make it be a conditionally normal one. Here, we focus on a different distributional class of GARCH innovation, namely the class of infinitely divisible distributions. In the following, we present the setup and the main idea of our model.

Suppose \((\Omega, \mathcal{F}, \mathbb{P})\) is a given complete probability space, where \( \mathbb{P} \) is the statistical or data-generating probability measure. Here, the sample space \( \mathcal{F} \) represents the uncertainty in our financial model. Let \( \mathcal{T} \) be the time index set \( \{0, 1, 2, \ldots, T\} \) of our financial model such that all economic activities take place at each time point \( t \in \mathcal{T} \). We equip our probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with the information structure \( \Phi := \{\Phi_t\}_{t \in \mathcal{T}} \). That is, for each \( t \in \mathcal{T}, \Phi_t \) represents the information set of all market information up to and including time \( t \).
Let \( \{\xi_t\}_{t \in \mathcal{T}} \) denote a stochastic process defined on the sample space \((\Omega, \mathcal{F})\) taking values on the real line \(\mathbb{R}\), with \(\xi_0 = 0\), which represents the random fluctuations of the stock-price process. We assume that for each \(t \in \mathcal{T}\), the value of \(\xi_t\) is known given the information \(\Phi_t\) up to and including time \(t\). Denote by \(\{h_t\}_{t \in \mathcal{T}}\) a conditional variance process of the underlying stock. We suppose that the value of \(h_t\) is known given the information \(\Phi_{t-1}\) up to and including time \(t-1\), for each \(t \in \mathcal{T} \setminus \{0\}\). Now, we further assume that \(\{\xi_t\}_{t \in \mathcal{T}}\) follows a GARCH process with orders \(p\) and \(q\) (GARCH\((p, q)\)) under the statistical probability measure \(\mathbb{P}\). More precisely, under \(\mathbb{P}\),

1. For each \(t \in \mathcal{T} \setminus \{0\}\), \(\xi_t|\Phi_{t-1} \sim F(0, h_t)\), where \(F(0, h_t)\) represents an infinitely divisible distribution with mean zero and conditional variance \(h_t\).

2. For each \(t \in \mathcal{T} \setminus \{0\}\),

\[
h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i \xi_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j},
\]

where \(p \geq 1, q \geq 1\) and \(\alpha_0 > 0, \alpha_i \geq 0, i \in \{1, 2, \ldots, p\}, \beta_j \geq 0, j \in \{1, 2, \ldots, q\}\) in order to ensure the positivity of \(h_t\).

Here, we relax the normality assumption for the GARCH innovation and consider the case that the innovation follows an infinitely divisible distribution. It has been noted in Tong (1990) that one direction of generalization of the ARCH-type models is to drop the normality requirement of the innovations process. See also Shephard (1996) for the discussion of the non-normal innovations for the ARCH-type models. Fan and Yao (2003) discussed some common types of GARCH models with non-Gaussian innovations, including the GARCH models with student-t innovations and the GARCH models with generalized Gaussian innovations. Both student-t distributions and generalized Gaussian distributions have heavier tails than normal distributions. Mills (1999) provided a comprehensive overview on different modifications of GARCH models with non-normal innovation. The proliferation of different GARCH models with non-normal innovation includes, for instances, the GARCH model with standardised-t innovation by Bollerslev (1987), the GARCH model with generalized exponential innovation by Nelson (1991),
etc. Like most of the ARCH-type models, our GARCH models are not time-reversible in general. That is, the probabilistic properties of the model are different when it is investigated backward through time. Most of the linear time series models with Gaussian innovations, for instances Gaussian AR(1) process, are time-reversible. Tong (1990) and Tong and Zhang (2003) provided an excellent exposition of time-reversibility of non-linear time series models in a general context. Taylor (1994) first proposed the use of the idea of time-reversible stochastic processes for distinguishing financial time series models.

For ensuring covariance stationarity of the GARCH(p, q) model, we further impose the condition that

\[ \sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1. \]

Let \( r \) be the constant continuously compounded risk-free interest rate of the bond \( B \) and \( \lambda \) the constant unit risk premium representing a preference parameter. Then, we assume that, under \( \mathbb{P} \), the dynamics of the bond-price process \( \{B_t\}_{t \in T} \) and the stock-price process \( \{S_t\}_{t \in T} \) satisfy:

\[
\begin{align*}
B_t &= B_{t-1} e^r, \quad B_0 = 1, \\
S_t &= S_{t-1} \exp(r + \lambda \sqrt{h_t} - \frac{1}{2} h_t + \xi_t), \quad S_0 = s, \quad t \in T \setminus \{0\}.
\end{align*}
\]

For each \( t \in T \setminus \{0\} \), \( Y_t \) denotes the continuously compounded one-period rate of return \( \ln \left( \frac{S_t}{S_{t-1}} \right) \) of the stock \( S \). Then, we can see that, under \( \mathbb{P} \), the conditional distribution of \( Y_t \) given the information \( \Phi_{t-1} \) is the infinitely divisible distribution \( F(r + \lambda \sqrt{h_t} - \frac{1}{2} h_t, h_t) \) with conditional mean \( r + \lambda \sqrt{h_t} - \frac{1}{2} h_t \) and conditional variance \( h_t \). In the following, we construct conditional Esscher transforms for the GARCH process \( \{Y_t\}_{t \in T} \) associated with a sequence of conditional Esscher parameters \( \{\theta_t\}_{t \in T} \).

First, suppose \( \{\theta_t\}_{t \in T \setminus \{0\}} \) is a stochastic process with the property that the value of \( \theta_t \) is known given the information \( \Phi_{t-1} \) up to and including time \( t - 1 \), for each \( t \in T \setminus \{0\} \). Let \( M_{Y_t|\Phi_{t-1}}(z) \) be the moment generating function of the conditional distribution \( Y_t \) given
of the statistical measure \( \Phi \) under the measure function \( F \) where \( B \) is an open interval on the real line and \( \{ Y_t \} \) is the conditional distribution function of \( Y_t \).

For each \( t \in T \setminus \{0\} \), we say that the moment generating function \( M_{Y_t|\Phi_{t-1}}(z) \) exists at point \( z \) if \( E_{\mathbb{P}}(e^{zY_t}|\Phi_{t-1}) < \infty \). Assume that \( M_{Y_t|\Phi_{t-1}}(\theta) \) exists, for all \( t \in T \). As in Bühlmann et al. (1996), we define a sequence \( \{ \Lambda_t \}_{t \in T} \) with \( \Lambda_0 = 1 \) and

\[
\Lambda_t = \prod_{k=1}^{t} \frac{e^{\theta_k Y_k}}{M_{Y_k|\Phi_{k-1}}(\theta_k)}, \quad t \in T \setminus \{0\}. \tag{2.3}
\]

Then, it is easy to check that \( \{ \Lambda_t \}_{t \in T} \) is a martingale under the statistical probability measure \( \mathbb{P} \) with respect to the information structure \( \Phi \). Write \( \mathbb{P}_t \) for the restriction \( \mathbb{P}|_{\Phi_t} \) of the statistical measure \( \mathbb{P} \) on the information set \( \Phi_t \), for each \( t \in T \setminus \{0\} \), where \( \mathbb{P}_T = \mathbb{P} \). We define a family of probability measures \( \{ \mathbb{P}_{t,\Lambda_t} \}_{t \in T \setminus \{0\}} \) by the following conditional Esscher transform:

\[
\mathbb{P}_{t,\Lambda_t}(\{ Y_t \in B \}|\Phi_{t-1}) = E_{\mathbb{P}_t}\left( I\{ Y_t \in B \} \frac{e^{\theta_t Y_t}}{E_{\mathbb{P}_t}(e^{\theta_t Y_t}|\Phi_{t-1})}|\Phi_{t-1} \right), \tag{2.4}
\]

where \( B \) is an open interval on the real line and \( I\{ Y_t \in B \} \) represents the indicator function of the event \( \{ Y_t \in B \} \). By the martingale property of the process \( \{ \Lambda_t \}_{t \in T} \) under \( \mathbb{P} \), we can show that \( \mathbb{P}_{t,\Lambda_t} = \mathbb{P}_{t+1,\Lambda_{t+1}}|\Phi_t \), for each \( t \in T \setminus \{T\} \).

The associated parameter \( \theta_t \) is called the conditional Esscher parameter given the information set \( \Phi_{t-1} \). Write \( F(y;\theta_t|\Phi_{t-1}) \) for the probability distribution of \( Y_t \) given \( \Phi_{t-1} \) under the measure \( \mathbb{P}_{t,\Lambda_t} \). From (2.4), \( F(y;\theta_t|\Phi_{t-1}) \) is given by:

\[
F(y;\theta_t|\Phi_{t-1}) = \frac{\int_{-\infty}^{y} e^{\theta_t x} dF(x|r + \lambda \sqrt{h_t} - \frac{1}{2} h_t, h_t)}{M_{Y_t|\Phi_{t-1}}(\theta_t)} \tag{2.5}
\]

Let \( M_{Y_t|\Phi_{t-1}}(z;\theta_t) \) denote the moment generating function of the adjusted distribution function \( F(y;\theta_t|\Phi_{t-1}) \). Then, we can see that \( M_{Y_t|\Phi_{t-1}}(z;\theta_t) \) can be related to \( M_{Y_t|\Phi_{t-1}}(z) \)
as follows.

\[ M_{Y_t|\Phi_{t-1}}(z; \theta_t) = \frac{M_{Y_t|\Phi_{t-1}}(z + \theta_t)}{M_{Y_t|\Phi_{t-1}}(\theta_t)}. \]  

(2.6)

For pricing the derivative \( V \), we construct a martingale pricing probability measure \( Q \) equivalent to the statistical probability measure \( P \) on the sample space \((\Omega, \mathcal{F})\) by adopting the concept of conditional Esscher transforms. First, we choose a sequence of conditional Esscher parameters \( \{\theta^q_t\}_{t \in T \setminus \{0\}} \) according to the following set of equations:

\[ r = \ln\{M_{Y_t|\Phi_{t-1}}(1; \theta^q_t)\}, \quad t \in T \setminus \{0\}. \]  

(2.7)

We can define a family of probability measures \( \{P_{t, \Lambda^q_t}\}_{t \in T \setminus \{0\}} \) associated with \( \{\theta^q_t\}_{t \in T \setminus \{0\}} \) by (2.4) with \( \{\theta_t\} \) replaced by \( \{\theta^q_t\} \).

Again, it can be checked that the family \( \{P_{t, \Lambda^q_t}\}_{t \in T \setminus \{0\}} \) satisfies the following consistency property:

\[ P_{t, \Lambda^q_t} = P_{s, \Lambda^q_s}|\Phi_t, \quad s, t \in T \text{ with } t \leq s. \]  

(2.8)

Let \( Q = P_{T, \Lambda^q_T} \) be a probability measure on \( \mathcal{F} = \Phi_T \). Then, we have the following proposition for the discounted stock-price process \( \{e^{-rt}S_t\}_{t \in T} \):

**Proposition 2.1:** The discounted stock-price process \( \{e^{-rt}S_t\}_{t \in T} \) is a martingale under \( Q \) with respect to the information structure \( \Phi \).

**Proof:** First, we show that for each \( t \in T \setminus \{0\} \),

\[ S_{t-1} = E_{P_{t, \Lambda^q_t}}(e^{-r}S_t|\Phi_{t-1}). \]  

(2.9)

Since the conditional distribution of \( Y_t \) given \( \Phi_{t-1} \) under the measure \( P_{t, \Lambda^q_t} \) is \( F(y; \theta^q_t|\Phi_{t-1}) \), \( M_{Y_t|\Phi_{t-1}}(z; \theta^q_t) \) is the corresponding moment generating function of \( Y_t \) given \( \Phi_{t-1} \). Then, by (2.7), we have

\[
E_{P_{t, \Lambda^q_t}}(e^{-r}S_t|\Phi_{t-1}) = S_{t-1}e^{-r}E_{P_{t, \Lambda^q_t}}(e^{Y_t}|\Phi_{t-1}) \\
= S_{t-1}e^{-r}M_{Y_t|\Phi_{t-1}}(1; \theta^q_t) = S_{t-1}.
\]
By the consistency property (2.8), for any \( t_1, t_2, t_3 \in T \) with \( t_1 \leq t_2 \leq t_3 \),
\[
E_{P_{t_3, \Lambda^q_{t_3}}}(e^{-rt_2}S_{t_2}|\Phi_{t_1}) = E_{P_{t_2, \Lambda^q_{t_2}}}(e^{-rt_2}S_{t_2}|\Phi_{t_1}). \tag{2.10}
\]
Then, we are going to show that, for any \( u, t \in T \) with \( t < u \),
\[
E_Q(e^{-ru}S_u|\Phi_t) = e^{-rt}S_t, \quad \text{almost surely with respect to } P. \tag{2.11}
\]
By using double expectation formulas, (2.9) and (2.10) iteratively, we get, for any \( u, t \in T \) with \( t < u \),
\[
E_Q(e^{-ru}S_u|\Phi_t) = E_{P_{u, \Lambda^q_u}}(e^{-ru}S_u|\Phi_{u-1})|\Phi_t] = E_{P_{u-1, \Lambda^q_{u-1}}}(e^{-r(u-1)}S_{u-1}|\Phi_{u-1})|\Phi_t]
\]
\[
= \ldots = E_{P_{t+1, \Lambda^q_{t+1}}}(e^{-r(t+1)}S_{t+1}|\Phi_{t+1}) = e^{-rt}S_t, \quad \text{almost surely with respect to } P.
\]
Hence, the result follows.

The fundamental theorem of asset pricing was proposed by Harrison and Kreps (1979) and further developed by Harrison and Pliska (1981), Dybyig and Ross (1987) and Delbaen and Schachermayer (1994). It established the relationship between the absence of arbitrage opportunities in a market and the existence of an equivalent martingale pricing measure under which the asset price process discounted at the risk-free interest rate is a martingale. By adopting the equivalent martingale pricing measure \( Q \), the price for the derivative \( V \) at time \( t \in T \) is given by:
\[
V_t = E_Q(e^{-r(T-t)}V_T|\Phi_t). \tag{2.12}
\]
We call \( Q \) a conditional risk-neutralized Esscher pricing measure. As in Gerber-Shiu’s option-pricing model, we can justify our pricing result by solving a dynamic utility maximization problem of an economic agent.

Let \( \{\gamma_t\}_{t \in T} \) denote a stochastic process on \((\Omega, \mathcal{F})\), where the value of \( \gamma_t \) is known given the information \( \Phi_t \) up to and including time \( t \), for each \( t \in T \). We define the
following sequence of random utility functions \( \{u_t\}_{t \in T} \) on the real line associated with \( \{\gamma_t\}_{t \in T} \):

\[
u_t(x) = \begin{cases} 
\frac{x^{1-\gamma_t}}{1-\gamma_t} & \text{if } \gamma_t \neq 1, \\
\ln x & \text{if } \gamma_t = 1.
\end{cases}
\]

For each fixed \( x \in \mathbb{R} \), \( u_t(x) \) is known given the information \( \Phi_t \) up to and including time \( t \). \( u_t : \mathbb{R} \to \mathbb{R} \) represents a power utility function with parameter \( \gamma_t \), for each \( t \in T \). As in Gerber-Shiu’s option-pricing model, we assume that an economic agent makes financial decisions according to the sequence of utility functions \( \{u_t\}_{t \in T} \). In particular, the agent adjusts or decides dynamically the risk-averse parameter \( \gamma_t \) based on the information \( \Phi_t \) up to and including time \( t \), for each \( t \in T \). Following Gerber and Shiu (1994), we impose the following assumptions about the agent.

**Assumption 2.2:**

1. The agent has \( m_t \) units of stock \( S \) over the time horizon \([t, t+1)\), where \( m_t \) can be decided according to the information \( \Phi_t \) up to and including time \( t \), for each \( t \in T \setminus \{T\} \).

2. For each \( t \in T \), \( \tilde{V}_t \) is the agent’s price of the derivative \( V \) at time \( t \) with \( \tilde{V}_T = V_T \), such that it is optimal for the agent not to buy or sell any unit of the derivative \( V \) at time \( t \).

The second statement of Assumption 2.2 can be related to a dynamic version of the variational argument in actuarial science. See Gerber and Shiu (2000) for the applications of the variational argument to the problem of dynamic asset allocation. For each \( t \in T \setminus \{T\} \), we define the following conditional expected utility function \( H_t \) on the agent’s wealth at time \( t+1 \) given the information \( \Phi_t \) up to and including time \( t \):

\[
H_t(\eta_t) = E_T\{u_t(m_tS_{t+1} + \eta_t[\tilde{V}_{t+1} - e^{r}\tilde{V}_t])|\Phi_t\},
\]

where \( \eta_t \) represents the number of units of the derivative \( V \) held by the agent over the time horizon \([t, t+1)\), which is the only choice variable for the maximization of \( H_t \).
By adopting the approach used in Gerber and Shiu (1994) under our dynamic setting, we justify our pricing result in the following proposition.

**Proposition 2.2:** For each \( t \in T \setminus \{ T \} \), \( \tilde{V}_t = V_t \).

**Proof:** The idea of the proof is similar to that in Gerber and Shiu (1994). First, by translating the second assumption in Assumption 2.2 mathematically, we have that \( H_t(\eta_t) \) attains its maximum value when \( \eta_t = 0 \), for each \( t \in T \setminus \{ 0 \} \). This implies that

\[
H'_t(0) = 0,
\tag{2.14}
\]

where \( H'_t(\eta_t) \) is the derivative of \( H_t(\eta_t) \) with respect to \( \eta_t \).

From (2.14), the price process \( \{ \tilde{V}_t \}_{t \in T} \) of the agent for the derivative \( V \) satisfies the following recursive equations:

\[
\tilde{V}_t = e^{-r} \frac{E_P(\tilde{V}_{t+1} S_{t+1}^{-\gamma_t} | \Phi_t)}{E_P(S_{t+1}^{-\gamma_t} | \Phi_t)} , \quad t \in T \setminus \{ T \} .
\tag{2.15}
\]

In fact, for all derivative securities, their corresponding agent’s price processes must satisfy the recursive relation (2.15). In particular, if we consider the trivial derivative instrument, namely the stock \( S \), the recursion (2.15) becomes

\[
S_t = e^{-r} \frac{E_P(S_{t+1}^{1-\gamma_t} | \Phi_t)}{E_P(S_{t+1}^{-\gamma_t} | \Phi_t)} , \quad t \in T \setminus \{ T \} .
\tag{2.16}
\]

From (2.16), we get

\[
S_t = e^{-r} S_t \frac{E_P \{ \exp((1 - \gamma_t) Y_{t+1}) | \Phi_t \}}{E_P \{ \exp(-\gamma_t Y_{t+1}) | \Phi_t \}} = e^{-r} S_t \frac{E_P \{ \exp((1 - \gamma_t) Y_{t+1}) | \Phi_t \}}{E_P \{ \exp(-\gamma_t Y_{t+1}) | \Phi_t \}} = e^{-r} S_t \frac{M_{Y_{t+1}|\Phi_t}(1 - \gamma_t)}{M_{Y_{t+1}|\Phi_t}(-\gamma_t)} , \quad t \in T \setminus \{ T \} .
\tag{2.17}
\]

Hence, by (2.6), (2.7) and (2.17), we have

\[
M_{Y_{t+1}|\Phi_t}(1; \theta_{t+1}) = e^r = \frac{M_{Y_{t+1}|\Phi_t}(1 - \gamma_t)}{M_{Y_{t+1}|\Phi_t}(-\gamma_t)}
\]
By the uniqueness of the conditional Esscher parameter, for each \( t \in T \setminus \{T\} \),

\[
\gamma_t = -\theta_{t+1}^q.
\]

(2.19)

Therefore, by (2.19) and Bayes’ rule, the recursive equation (2.15) can be written as

\[
\tilde{V}_t = e^{-r} \frac{E_p(\tilde{V}_{t+1} S_{t+1} \theta_{t+1}^q | \Phi_t)}{E_p(\tilde{S}_{t+1} \theta_{t+1}^q | \Phi_t)} = e^{-r} \frac{E_p(\tilde{V}_{t+1} \exp(\theta_{t+1}^q Y_{t+1}) | \Phi_t)}{E_p(\exp(\theta_{t+1}^q Y_{t+1}) | \Phi_t)}
\]

\[
= e^{-r} \frac{E_p(\tilde{V}_{t+1} \exp(\theta_{t+1}^q Y_{t+1}) | \Phi_t)}{E_p(\exp(\theta_{t+1}^q Y_{t+1}) | \Phi_t)} = e^{-r} E_{p_{t+1, \Lambda_{t+1}^q}}(\tilde{V}_{t+1} | \Phi_t).
\]

(2.20)

As in Proposition 2.1, we use (2.20) and the consistency property (2.8) to get

\[
\tilde{V}_t = e^{-r} E_{p_{t+1, \Lambda_{t+1}^q}}(\tilde{V}_{t+1} | \Phi_t) = e^{-r(T-t)} E_{q_{T, \Lambda_T^q}}(V_T | \Phi_t)
\]

\[
= e^{-r(T-t)} E_Q(V_T | \Phi_t) = V_t, \quad t \in T.
\]

(2.21)

□

Proposition 2.2 states that the economic agent’s price process \( \{\tilde{V}_t\}_{t \in T} \) coincides with our price process \( \{V_t\}_{t \in T} \) in order to give the agent no incentive to buy or sell any fraction or multiple of the derivative \( V \) at any time point \( t \in T \).

§3. Some Parametric Cases

In the following, we consider two interesting cases of our model by imposing parametric assumptions on the conditional distribution of the innovation \( \xi_t \) given \( \Phi_{t-1} \). First, we consider the case of conditional normal distribution for the GARCH innovation. Then, we describe in some detail the case of the conditional shifted gamma distribution for the GARCH innovation.

§3.1. Conditional Normality for the Innovation

\[
= M_{Y_{t+1} \Phi_t}(1; -\gamma_t), \quad t \in T \setminus \{T\}.
\]

(2.18)
We consider the case that the conditional distribution of the noise $\xi_t$ given $\Phi_{t-1}$ follows a normal distribution with conditional mean zero and conditional variance $h_t$ under the statistical probability measure $\mathbb{P}$. In this case, we can also obtain a formula for the conditional risk-neutralized Esscher parameter $\theta^q_t$, for each $t \in \mathcal{T}\setminus\{0\}$.

Following Duan (1995), we assume that $\{\xi_t\}_{t \in \mathcal{T}}$ follows a GARCH ($p$, $q$) process with the conditional distribution $F(0, h_t)$ given $\Phi_{t-1}$ being a normal distribution with conditional mean zero and variance $h_t$ under statistical probability measure $\mathbb{P}$. Then, the conditional distribution of $Y_t$ given $\Phi_{t-1}$ under $\mathbb{P}$ is a normal distribution with mean $r + \lambda \sqrt{h_t} - \frac{1}{2} h_t$ and variance $h_t$. For each $t \in \mathcal{T}\setminus\{0\}$, the conditional risk-neutralized Esscher parameter $\theta^q_t$ is given by the following proposition.

**Proposition 3.1:** For each $t \in \mathcal{T}\setminus\{0\}$,

\[
\theta^q_t = -\frac{\lambda}{\sqrt{h_t}},
\]

and hence

\[
\gamma_{t-1} = \frac{\lambda}{\sqrt{h_t}}.
\]

**Proof:** By (2.6) and (2.7), we have

\[
\begin{align*}
r &= \ln\{M_{Y_t|\Phi_{t-1}}(1; \theta^q_t)\} = \ln\left\{ \frac{M_{Y_t|\Phi_{t-1}}(1 + \theta^q_t)}{M_{Y_t|\Phi_{t-1}}(\theta^q_t)} \right\} \\
&= \ln\left\{ \frac{\exp[(1 + \theta^q_t)(r + \lambda \sqrt{h_t} - \frac{1}{2} h_t) + \frac{1}{2}(1 + \theta^q_t)^2 h_t]}{\exp[\theta^q_t(r + \lambda \sqrt{h_t} - \frac{1}{2} h_t) + \frac{1}{2}\theta^q_t^2 h_t]} \right\} \\
&= r + \lambda \sqrt{h_t} + \theta^q_t h_t, \quad t \in \mathcal{T}\setminus\{0\}.
\end{align*}
\]

This implies the results (3.1) and (3.2) follows immediately from (2.19).

As time goes by, the evolution of $\{\theta^q_t\}_{t \in \mathcal{T}\setminus\{0\}}$ can capture the changing nature of the conditional variance $h_t$. Then, by (2.6) and (3.1), the moment generating function $M_{Y_t|\Phi_{t-1}}(z; \theta^q_t)$ is given by

\[
M_{Y_t|\Phi_{t-1}}(z; \theta^q_t) = \frac{M_{Y_t|\Phi_{t-1}}(z + \theta^q_t)}{M_{Y_t|\Phi_{t-1}}(\theta^q_t)}
\]
\[
\frac{\exp[(z + \theta_i^q)(r + \lambda \sqrt{h_t} - \frac{1}{2} h_t) + \frac{1}{2}(z + \theta_i^q)^2 h_t]}{\exp[\theta_i^q(r + \lambda \sqrt{h_t} - \frac{1}{2} h_t) + \frac{1}{2} \theta_i^q h_t]}
= \exp[z(r - \frac{1}{2} h_t) + \frac{1}{2} z^2 h_t].
\]

(3.3)

Hence, under \(Q\), the conditional distribution of \(Y_t\) given \(\Phi_{t-1}\) is a normal distribution with conditional mean \(r - \frac{1}{2} h_t\) and conditional variance \(h_t\), where \(h_t\) is given by

\[
h_t = \alpha_0 + \sum_{i=1}^p \alpha_i \xi_{t-i}^2 + \sum_{j=1}^q \beta_j h_{t-j}, \quad t \in T \setminus \{0\}.
\]

It is important to note that the conditional distribution of \(\xi_t\) given \(\Phi_{t-1}\) is a normal distribution with mean \(-\lambda \sqrt{h_t}\) and variance \(h_t\) under the martingale pricing probability measure \(Q\), for each \(t \in T \setminus \{0\}\). \(^3\) Define a new random variable \(\epsilon_t := \xi_t + \lambda \sqrt{h_t}\), for each \(t \in T \setminus \{0\}\). Then, under \(Q\), the conditional distribution of \(\epsilon_t\) given \(\Phi_{t-1}\) is a normal distribution with mean zero and variance \(h_t\). Thus, under \(Q\), we can write the dynamic of the underlying stock-price process in the following form:

\[
Y_t = r - \frac{1}{2} h_t + \epsilon_t,
\]

where the conditional variance process is given by:

\[
h_t = \alpha_0 + \sum_{i=1}^p \alpha_i (\epsilon_{t-i} - \lambda \sqrt{h_{t-i}})^2 + \sum_{j=1}^q \beta_j h_{t-j}.
\]

The aforementioned underlying asset’s price process and the conditional variance process under \(Q\) is the same as those in Duan (1995). Hence, our pricing result in this case is in complete agreement with that of Duan (1995).

\section*{3.2. Heston and Nandi’s GARCH Option Pricing Model}

We consider the GARCH option pricing model proposed in the paper by Heston and Nandi (2000). The conditional Esscher transform provides a way to pick an equivalent...
martingale pricing measure $\mathbb{Q}$ under which the risk-neutral GARCH process for the dynamic of the underlying stock price is the same as that in Heston and Nandi (2000). For each $t \in T \setminus \{0\}$, we define a standardized normal random variable $z_t$ under the statistical probability measure $\mathbb{P}$. In the context of Heston and Nandi’s GARCH option pricing model, the dynamic of the logarithmic return process of the underlying stock $S$ under $\mathbb{P}$ is governed by:

$$Y_t = r + \lambda h_t + \sqrt{h_t}z_t, \quad S_0 = s, \ t \in T \setminus \{0\},$$

and conditional variance process is given by:

$$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i(z_{t-i} - \rho_i \sqrt{h_{t-i}})^2 + \sum_{j=1}^{q} \beta_j h_{t-j},$$

where the modulus of the roots of the following characteristic polynomial must be less than one:

$$P(z) = z^q - \sum_{i=1}^{q} (\beta_i + \alpha_i \rho_i^2) z^{q-i}.$$

For each $t \in T \setminus \{0\}$, the conditional distribution of $Y_t$ given $\Phi_{t-1}$ is a normal distribution with conditional mean being $r + \lambda h_t$ and conditional variance being $h_t$. In this case, the conditional risk-neutralized Esscher parameter $\theta^q_t$ is given by:

**Proposition 3.2:** For each $t \in T \setminus \{0\}$,

$$\theta^q_t = -\left(\lambda + \frac{1}{2}\right),$$

and hence

$$\gamma_{t-1} = \lambda + \frac{1}{2}.$$

**Proof:** The proof of Proposition 3.2 is adapted to that of Proposition 3.1.}

From Proposition 3.2, both the risk-neutralized Esscher parameter $\theta^q_t$ and the risk-averse parameter $\gamma_{t-1}$ of the power utility function only depend on the parameter of unit risk premium $\lambda$, but do not depend on the conditional variance $h_t$, for each $t \in T \setminus \{0\}$. 

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Note that the conditional distribution of $Y_t$ given $\Phi_{t-1}$ is a normal distribution with conditional mean $r - \frac{1}{2} h_t$ and conditional variance $h_t$ under the risk-neutralized Esscher measure $Q$. For each $t \in T \setminus \{0\}$, define $z_t^q := z_t + (\lambda + \frac{1}{2}) \sqrt{h_t}$. Then, $z_t^q \sim N(0, 1)$ under $Q$. The dynamic of the logarithmic return process of the underlying stock $S$ under $Q$ is given by:

$$Y_t = r - \frac{1}{2} h_t + \sqrt{h_t} z_t^q, \ t \in T \setminus \{0\},$$

and the conditional variance process is governed by:

$$h_t = \alpha_0 + \sum_{i=1}^p \alpha_i (z_{t-i}^q - (\lambda + \frac{1}{2} + \rho_i) \sqrt{h_{t-i}})^2 + \sum_{j=1}^q \beta_j h_{t-j}.$$ 

The risk-neutral dynamic in this case is exactly the same as that of Heston and Nandi (2000).

### §3.3. Conditional Shifted Gamma Distribution for the Innovation

In this subsection, we deal with the case of the conditional shifted gamma distribution for the innovations process. The GARCH model with shifted gamma innovation provides practitioners with the flexibility of modelling the skewed behavior in the stock innovation.

First, we consider a stochastic process $\{X_t\}_{t \in T \setminus \{0\}}$ such that under the statistical measure $P$,

1. For each $t \in T \setminus \{0\}$, the value of $X_t$ is known given the information $\Phi_t$ up to and including time $t$.
2. $\{X_t\}_{t \in T \setminus \{0\}}$ are independent and identically distributed.
3. For each $t \in T \setminus \{0\}$, $X_t \sim Ga(a, b)$, where $Ga(a, b)$ represents a gamma distribution with shape parameter $a$ and scale parameter $b$. That is, the density function $f_{X_t}(x)$ of $X_t$ is given by:

$$f_{X_t}(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}, \ t \in T \setminus \{0\}. \quad (3.6)$$
Then, we assume that the stock innovation $\xi_t$ is given as follows:

$$
\xi_t = \sqrt{h_t} \left( \frac{X_t - a}{\sqrt{h_t}} \right), \ t \in T \setminus \{0\}. 
$$

(3.7)

Hence, under $\mathbb{P}$, the conditional distribution of $\xi_t$ given $\Phi_{t-1}$ is a shifted gamma distribution with mean zero and variance $h_t$. That is,

$$
\xi_t | \Phi_{t-1} \sim \text{SGa}(0, h_t) , \ t \in T \setminus \{0\}. 
$$

(3.8)

In this case, we have the following specifications for the logarithmic stock-price returns process $\{Y_t\}_{t \in T \setminus \{0\}}$ and the conditional variance process $\{h_t\}_{t \in T \setminus \{0\}}$.

1. $Y_t = r + \lambda \sqrt{h_t} - \frac{1}{2} h_t + \xi_t$, where the conditional distribution $F(0, h_t)$ of $\xi_t$ given $\Phi_{t-1}$ is $SGa(0, h_t)$.

2. $h_t = \alpha_0 + \sum_{i=1}^p \alpha_i \xi_{t-i}^2 + \sum_{j=1}^q \beta_j h_{t-j}$.

The conditional variance process is driven by a shifted gamma innovations under the $\mathbb{P}$-measure. In the variance-gamma SV model proposed by Madan, Carr and Chang (1998), the dynamic of the stochastic volatility is also driven by a gamma innovations process. The main difference between our shifted gamma GARCH model and the variance-gamma SV model is that our conditional variance process is driven by one observable gamma innovations process while the volatility process of the variance-gamma model is unobservable and driven by an unobservable gamma innovations process.

Here, we can also write the dynamic of the logarithmic stock-price return process $\{Y_t\}_{t \in T \setminus \{0\}}$ under the $\mathbb{P}$-measure in terms of $\{X_t\}_{t \in T \setminus \{0\}}$ as follows.

$$
Y_t = r + \lambda \sqrt{h_t} - \frac{1}{2} h_t - \sqrt{ah_t} + b \sqrt{\frac{h_t}{a}} X_t , \ t \in T \setminus \{0\}. 
$$

(3.9)

By noticing that $b \sqrt{\frac{h_t}{a}} X_t | \Phi_{t-1} \sim Ga(a, \sqrt{\frac{h_t}{a}})$, the conditional distribution of $Y_t$ given $\Phi_{t-1}$ is a shifted gamma distribution with shape parameter $a$, scale parameter $\sqrt{\frac{h_t}{a}}$ and
shifted parameter $-r - \lambda \sqrt{h_t} + \frac{1}{2} h_t + \sqrt{ah_t}$. Hence, the moment generating function of $Y_t$ given $\Phi_{t-1}$ is given by

$$M_{Y_t|\Phi_{t-1}}(\theta) = \left( \frac{\sqrt{\frac{a}{h_t}}}{\sqrt{\frac{a}{h_t}} - \theta} \right)^a \exp[(r + \lambda \sqrt{h_t} - \frac{1}{2} h_t - \sqrt{ah_t})\theta], \quad \theta < \sqrt{\frac{a}{h_t}}. \quad (3.10)$$

The moment generating function $M_{Y_t|\Phi_{t-1}}(z; \theta_t)$ of $Y_t$ given $\Phi_{t-1}$ under the transformed probability measure $P_{t, \Lambda_t}$ is

$$M_{Y_t|\Phi_{t-1}}(z; \theta_t) = \left( \frac{\sqrt{\frac{a}{h_t}} - \theta_t}{\sqrt{\frac{a}{h_t}} - \theta_t - z} \right)^a \exp[(r + \lambda \sqrt{h_t} - \frac{1}{2} h_t - \sqrt{ah_t})z], \quad (3.11)$$

provided that $z < \sqrt{\frac{a}{h_t}} - \theta_t$.

By (2.7) and (3.12), we get an explicit formula for the conditional risk-neutralized Esscher parameter as follows.

$$\theta_t^q = \sqrt{\frac{a}{h_t}} - \left[ 1 - \exp\left( \frac{\lambda \sqrt{h_t} - \frac{1}{2} h_t - \sqrt{ah_t}}{a} \right) \right]^{-1}. \quad (3.12)$$

Write $b_t$ for the scale parameter $\sqrt{\frac{a}{h_t}}$ of the shifted gamma distribution under the statistical measure $P$. Then, we define an $\Phi_{t-1}$-measurable parameter $b_t^q$ as follows.

$$b_t^q := b_t - \theta_t^q = \left[ 1 - \exp\left( \frac{\lambda \sqrt{h_t} - \frac{1}{2} h_t - \sqrt{ah_t}}{a} \right) \right]^{-1}, \quad t \in T \setminus \{0\}. \quad (3.13)$$

From (3.12) and (3.14), we see that under $Q$,

$$Y_t|\Phi_{t-1} \sim SGa(a, b_t^q; -r - \lambda \sqrt{h_t} + \frac{1}{2} h_t + \sqrt{ah_t}), \quad t \in T \setminus \{0\}. \quad (3.14)$$

Unlike the conditional normality case, the relationship among the conditional risk-neutralized Esscher parameter $\theta_t^q$, the constant unit risk premium $\lambda$ and the conditional risk-averse parameter $\gamma_t$ becomes more obscured in this case. However, as in Gerber-Shiu’s option-pricing model, the conditional distribution of $Y_t$ given $\Phi_{t-1}$ under the $Q$-measure...
is still a shifted gamma distribution. In fact, the conditional risk-neutralized Esscher transform only changes the real-world scale parameter $b_t$ of the original shifted gamma distribution under the $\mathbb{P}$-measure to a conditional risk-neutralized scale parameter $b^q_t$. Then, we can also write the dynamic of the logarithmic stock-price returns under the $\mathbb{Q}$-measure in the following form:

$$Y_t = r + \lambda \sqrt{h_t} - \frac{1}{2} h_t - \sqrt{ah_t} + X^q_t, \ t \in T \setminus \{0\} , \ (3.15)$$

where $X^q_t \sim Ga(a, b^q_t)$ and the process $\{h_t\}_{t \in T \setminus \{0\}}$ is governed by:

$$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i (X^q_{t-i} - \sqrt{ah_{t-i}})^2 + \sum_{j=1}^{q} \beta_j h_{t-j}, \ t \in T \setminus \{0\}. \ (3.16)$$

We adopt (3.16) and (3.17) to simulate the price process of the underlying stock under the $\mathbb{Q}$-measure in Section 4. It is important to note that the conditional variance process $\{h_t\}_{t \in T \setminus \{0\}}$ under the $\mathbb{P}$-measure is no longer the conditional variance process for the price of the underlying stock under the $\mathbb{Q}$-measure. In fact, for each $t \in T \setminus \{0\}$, the conditional variance $h^q_t$ under the $\mathbb{Q}$-measure follows a non-linear GARCH (p,q) process. That is, the conditional variance $h^q_t$ is a non-linear function of past conditional variances and innovations under the $\mathbb{Q}$-measure. We can express $h^q_t$ in terms of $h_t$ as follows:

$$h^q_t = \frac{a^2}{b^q_t} = a^2 \left[ 1 - \exp \left( \frac{\lambda \sqrt{h_t} - \frac{1}{2} h_t - \sqrt{ah_t}}{a} \right) \right], \ t \in T \setminus \{0\}. \ (3.17)$$

Finally, it is interesting to mention that the parameter $b$ in the distribution $Ga(a, b)$ does not appear in the GARCH models for the underlying stock prices under the martingale pricing measure $\mathbb{Q}$. Hence, we do not need to estimate the parameter $b$ for our pricing purpose.

§4. Numerical Experiment

In this section, we present numerical results for the comparison of our GARCH option pricing model with the standard Black-Scholes model using the close values of S&P 500 daily index series from June 27, 1997 to June 27, 2003, a total of six-year data with 1507
observations. In particular, we focus on the case of conditional shifted gamma GARCH innovations. The data were obtained from the database in Yahoo finance.

It has been mentioned in Fan and Yao (2003) that the GARCH(1,1) model has been successful empirically and is considered the benchmark model for volatility modelling in financial econometrics. Here, we use the GARCH(1,1) model for illustration. As in Duan (1995), we assume that the risk-free interest rate $r$ is zero throughout this section. Monte Carlo simulations coupled with its variance reduction techniques, namely control-variate techniques, are adopted here to simulate from the GARCH models under the risk-neutralized Esscher measure. The seminal work by Boyle (1977) was the pioneer on the use of Monte Carlo simulation and its control-variate techniques to compute option prices. Here, we adopt the control-variate techniques based on the theoretical and simulated Gerber-Shiu’s option prices for the simulation of the GARCH option prices. We will consider standard European call options written on the S&P 500 index. We perform fifty thousands simulation runs for a given set of GARCH option prices. All the computations in this section were done by Fortran codes with IMSL subroutines.

Tong (1990) mentioned that there are basically three approaches to the estimation of parametric non-linear time series models, namely the maximum likelihood approach, the conditional least-squares approach and the method of moments approach. Fan and Yao (2003) discussed three types of estimators for the GARCH models, namely the conditional maximum likelihood estimator, the Whittle’s estimator and the least absolute deviation estimator. They mentioned that the conditional maximum likelihood estimator is one of the most popular methods in fitting GARCH models and has been widely used in the banking and finance industries. It enjoys some desirable sampling and asymptotic properties. It has also been mentioned in Franses and van Dijk (2000) that if one is not sure whether the specified parametric distribution assumption for the GARCH innovation is correct, one may resort to the Quasi-Maximum Likelihood Estimation (QMLE). It is an approximation for the exact MLE by replacing the likelihood of the specific distribution for the GARCH innovation with the normal likelihood. For an overview of various methods for the estimation of the ARCH-type models and non-linear time series models, refer to Franses and van Dijk (2000), Mills (1999), Shephard (1996), Tong (1990), Tong (2002),

Here, we adopt a two-stage estimation procedure for the estimation of the GARCH model with conditional shifted gamma innovations. At the first stage, the QMLE is used to estimate the GARCH parameters $\alpha_0, \alpha_1, \beta_1$ and $\lambda$. We summarize the Quasi-Maximum Likelihood Estimates in Table 4.1:

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Estimated parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_0$</td>
<td>$3.577 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>0.155966</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.646049</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$4.873824 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

At the second stage, we adopt the method of moments approach to estimate the unknown parameter $a$. Given the market data of the logarithmic returns of the underlying stock $\{Y_1, Y_2, \ldots, Y_N\}$ and the values of the estimated parameters $\hat{\alpha}_0, \hat{\alpha}_1, \hat{\beta}_1$ and $\hat{\lambda}$, we use both the realized time series $\{\xi_1, \xi_2, \ldots, \xi_N\}$ and $\{h_1, h_2, \ldots, h_N\}$ for the evaluation of the method of moments estimator. By using the method of moments approach and the unconditional third moment of the GARCH innovation, we provide the following formula for the estimator $\hat{a}$ of the parameter $a$:

$$\hat{a} = \left[ \frac{2(\sum_{t=1}^{N} h_t^{3/2})}{\sum_{t=1}^{N} \xi_t^3} \right]^2. \quad (3.18)$$

From the S&P 500 daily index series, the numerical value of the estimator $\hat{a}$ is 187.826128.
Table 4.2 presents the Black-Scholes call prices and the call prices obtained from the GARCH model with conditional shifted gamma innovations corresponding to different price-to-strike ratios ($S/K$), time to maturities $TM$ and initial conditional variance $IV$. The current closing value of the index is 976.22.

Table 4.2. Comparison between Black-Scholes call prices and the call prices obtained by the GARCH model with conditional shifted gamma innovations

<table>
<thead>
<tr>
<th>Maturity (days)</th>
<th>$S/K$</th>
<th>B-S</th>
<th>GARCH-SG $IV = 0.8$</th>
<th>GARCH-SG $IV = 1.2$</th>
<th>GARCH-SG $IV = 1.6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TM = 30$</td>
<td>1.00</td>
<td>28.6658</td>
<td>27.2371</td>
<td>28.4561</td>
<td>30.0598</td>
</tr>
<tr>
<td></td>
<td>1.05</td>
<td>57.1521</td>
<td>56.0249</td>
<td>56.9504</td>
<td>58.2053</td>
</tr>
<tr>
<td></td>
<td>1.10</td>
<td>91.9016</td>
<td>91.4053</td>
<td>91.8642</td>
<td>92.5364</td>
</tr>
<tr>
<td></td>
<td>1.15</td>
<td>128.0766</td>
<td>127.9168</td>
<td>128.0686</td>
<td>128.3314</td>
</tr>
<tr>
<td>$TM = 90$</td>
<td>1.00</td>
<td>49.6281</td>
<td>48.6508</td>
<td>49.3304</td>
<td>50.2688</td>
</tr>
<tr>
<td></td>
<td>1.05</td>
<td>75.1878</td>
<td>74.1654</td>
<td>74.7777</td>
<td>75.6210</td>
</tr>
<tr>
<td></td>
<td>1.10</td>
<td>104.3662</td>
<td>103.5147</td>
<td>103.9947</td>
<td>104.6666</td>
</tr>
<tr>
<td></td>
<td>1.15</td>
<td>135.3505</td>
<td>134.7303</td>
<td>135.0597</td>
<td>135.5269</td>
</tr>
<tr>
<td>$TM = 180$</td>
<td>1.00</td>
<td>70.1373</td>
<td>68.7495</td>
<td>69.1972</td>
<td>69.8343</td>
</tr>
<tr>
<td></td>
<td>1.05</td>
<td>94.1940</td>
<td>92.8174</td>
<td>93.2356</td>
<td>93.8301</td>
</tr>
<tr>
<td></td>
<td>1.10</td>
<td>120.4283</td>
<td>119.1510</td>
<td>119.5108</td>
<td>120.0236</td>
</tr>
<tr>
<td></td>
<td>1.15</td>
<td>147.9018</td>
<td>146.7640</td>
<td>147.0504</td>
<td>147.4624</td>
</tr>
</tbody>
</table>

It is well documented that the Black-Scholes model always underprices deep out-of-the-money options and short-maturity options (see Black (1975) and Whaley (1982)). Duan (1995) provided a very comprehensive comparison between the GARCH option prices obtained under the conditional normal innovations and the Black-Scholes option prices using the $S&P100$ index options. His numerical results revealed that compared...
with the GARCH option prices, the Black-Scholes model always underprice deep out-of-the-money options and it can underprice or overprice an out-of-the-money options depending on the level of the initial conditional volatility. It has also been shown in his numerical results that the underpricing effect of the Black-Scholes model is magnified when the maturity of the option become shorter. His numerical results provide an important empirical justification for the flexibility of the GARCH option pricing model in pricing options by incorporating changing conditional variances. The numerical results presented in Table 4.2 are consistent with those in Duan (1995). In particular, the Black-Scholes model always underprice deep out-of-the-money options and the underpricing effect is more pronounced when the maturity of the option becomes shorter or when the level of the initial conditional volatility becomes higher. We can also see from Table 4.2 that compared with the GARCH option prices, the underpricing or overpricing of the Black-Scholes model depends on the level of the initial conditional volatility. This illustrates the flexibility our GARCH model for valuing options with the effect of changing conditional volatilities being incorporated.

§5. Conclusion and Further Research

We introduced an option-pricing model under the GARCH assumption for the underlying stock-price dynamic. The concept of conditional Esscher transforms has been adopted to pick a particular pricing probability measure under the incomplete market induced by both the discrete-time and continuous-state nature of the GARCH process. Our pricing result can be justified by the dynamic utility framework which provides an intuitive economic argument for market equilibrium in our discrete-time economy.

One possible direction of investigation is to explore the applications of our model to different types of underlying financial instruments which exhibit deviation of the conditional non-normality assumption of the GARCH innovation. We may also consider the extension of our model to deal with American-style options. For other extensions of our model, we may consider the option-pricing model under different types of GARCH specifications, for instances, exponential GARCH (EGARCH) and threshold GARCH (T-GARCH). Härdle and Hafner (2000) and Tong (1990) may provide some clues for the
development of our model in this direction. Applications of our model to price various kinds of reinsurance products which exhibit “derivative” feature are a topic of practical interest.

The class of GARCH models with infinitely divisible distributions for their innovations also presents some interesting problems for further investigation. It may be interesting to develop other more efficient estimation procedures for our GARCH models and investigate the sampling properties of the estimators for our GARCH models and their corresponding asymptotic properties. It may be interesting to investigate the semi-parametric and non-parametric methods for the estimation of GARCH models with infinitely divisible distributions; that is, we let the data speak for themselves which parametric form of the distribution within the class of the infinitely divisible distributions is appropriate for the GARCH innovation. Fan and Yao (2002) and Tong (2002) may provide some clues for approaching this challenging problem.

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References


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