Abstract

This paper proposes a model for measuring risks for derivatives which is easy to implement and satisfies a set of four coherent properties introduced in Artzner et al. (1999). We construct our model within the context of Gerber-Shiu’s option-pricing framework. A new concept, namely Bayesian Esscher “scenarios,” which generalizes the concept of generalized “scenarios,” is introduced via a “Random Esscher Transform.” Our risk measure involves the use of the risk-neutral Bayesian Esscher “scenario” for pricing and a family of real-world Bayesian Esscher “scenarios” for risk measurement. Closed-form expressions for our risk measure can be obtained in some special cases.
Key words/phrases: Coherent risk measures, Bayesian Esscher “scenarios,” Random Esscher measures, Mixture model, Non-linear risk, Credibility theory.
§1. Introduction

Risk management is an important issue in finance and actuarial science. Investment banks, financial corporations and insurance companies around the globe are searching for techniques to enhance their risk management practice. Due to the rapid development of derivative markets, the practice becomes more complex and challenging. This accelerates the development of more advanced techniques in risk management and creates many interesting theoretical and practical problems for risk researchers. One of the critical steps in managing risks is to construct a proper measure of risk. Traditionally, ruin probability is a standard risk measure in actuarial science while volatility is a commonly used risk measure in the finance community. Recently, Value-at-Risk (VaR) has become a very popular measure of risk. VaR is an attempt to summarize the total risk of a portfolio by a single number which is a statistical estimation of a portfolio loss with the property that, with a given (small) probability, the owner of the portfolio stands to incur that loss or more over a given (typically short) holding period. See Embrechts (2000) and J.P. Morgan’s Risk Metrics-Technical Document for an introduction and Duffie and Pan (1997) for a survey. Artzner et al. (1999) proposed four desirable properties for risk measures, namely translation invariance, positive homogeneity, monotonicity and subadditivity. A risk measure satisfying these four properties is called a coherent risk measure. They pointed out that VaR does not, in general, satisfy the subadditivity property, especially if the portfolio contains derivatives which may make the portfolio’s distribution non-standard (or, more precisely, non-elliptical). They also proved that a coherent risk measure can be represented by the supremum of the expectations of the portfolio’s change over the holding period with respect to a family of generalized “scenarios” or probability distributions. Any coherent risk measure can be generated from the representation form by choosing an appropriate set of generalized “scenarios.” Wang et al. (1997) developed axioms similar to those of Artzner et al., but in an insurance context.

Measuring risks of non-linear instruments, such as derivatives, is a complex and challenging problem for risk researchers. It is difficult to find an unifying approach for measuring risks of derivatives which is easy to implement and satisfies some theo-
retically consistent properties. Traditional methods of measuring and managing risks of derivatives rely solely on the so-called Greek letters, namely delta, gamma, theta, vega, etc. Since the introduction of VaR, much effort has been placed on investigating VaR for derivatives. See, for example, J.P. Morgan’s Risk Metrics-Technical Document, Duffie and Pan (1997) and Jahel et al. (1999). Basically, there are two common analytical approaches to the calculation of VaR for derivatives, namely the delta-gamma approach and the method of matching portfolio’s moments. Both approaches are difficult to implement if a portfolio consists of a significant number of non-linear instruments and complex derivatives. Sometimes, it is difficult to find a distribution which matches the portfolio’s moments well, and hence the method of matching portfolio’s moments is difficult to implement in a real situation. By the second “Fundamental theorem of risk management” (see Embrechts (2000)), VaR does not satisfy the subadditivity property if the portfolio’s distribution is non-elliptical. Hence, it is difficult to preserve the subadditivity property (hence coherence) of VaR for a portfolio of derivatives due to the non-elliptical nature of the portfolio’s distribution.

In this paper, we propose a coherent approach to measure risks of derivatives within the context of Gerber-Shiu’s option-pricing model (see Gerber and Shiu (1994)). Our approach is easy to implement and is flexible enough to preserve the coherence for measuring risks of a wide variety of derivatives under different parametric models on the stock-price dynamic, such as the Wiener process, the multiplicative binomial process, the Poisson process, the gamma process and the inverse-Gaussian process, and others. A new concept, namely the Bayesian Esscher “scenarios,” which further generalizes the concept of generalized “scenarios,” is introduced. The concept of Bayesian Esscher “scenarios” takes both an investor’s subjective views (or risk preferences) and the objective market observations (or data) into account in choosing a family of generalized “scenarios” or probability distributions for the stock-price movements. It is natural to relate the concept of Bayesian Esscher “scenarios” to that of credibility theory in actuarial science. It is also interesting to note that the concept of Bayesian Esscher “scenarios” is flexible enough to incorporate the pricing framework as a special case, called the risk-neutral Bayesian Esscher “scenario,” in order to make our framework consistent with Gerber-Shiu’s pricing model. By mod-
ifying the representation form of coherent risk measures introduced by Artzner et al. (1999), our risk measure is scenario-based and involves the use of the risk-neutral Bayesian Esscher “scenario” for pricing and a family of real-world Bayesian Esscher “scenarios” for measuring risks. Then, our risk measure is calculated as the supremum of the expected portfolio’s loss over the fixed time horizon with respect to a family of real-world Bayesian Esscher “scenarios.” Our risk measure can also be used to measure the risk inherent in “derivatives” in actuarial science, such as reinsurance treaties. For the sake of completeness, we need to include a minimal amount of probability theory. The level of mathematical rigor is then similar to that of Gerber and Shiu (1994). Now, our paper is organised as follows.

Section two presents the main idea of our approach. For illustration, we consider a simple financial model consisting of a risk-free bond and a risky stock. We investigate a coherent risk measure for a single European call option in order to provide some insights on the risk of the whole portfolio. Here, the risk of an unhedged position rather than the risk of incomplete/imperfect hedging is considered. The risk of incomplete hedging has been studied extensively by Cvitanic and Karatzas (1999), Föllmer and Leukert (1999) and Runggaldier and Zaccaria (2000), and others. Motivated by Gerber and Shiu (1994), Artzner et al. (1999) and Siu and Yang (1999), the concepts of “Random Esscher Transform” and Bayesian Esscher “scenarios” are introduced. Our risk measure is of practical relevance because of the rapid growth in the trading volume of derivatives in secondary markets. Section three considers some special cases of our approach. Closed-form expressions of our risk measure can be obtained in some special cases. The credibility interpretation becomes more transparent in some special cases. Finally, we conclude our paper with suggestions of some possible directions for further research.

§2. Main idea of our approach

In this section, we present a coherent approach for measuring risk inherent in an unhedged position of derivatives. First, we suppose that our financial model consists of a risk-free bond $B$ and a risky underlying asset $S$. For illustration, we focus on a coherent risk measure for an unhedged position of a call option $C$ written on $S$
over a fixed time horizon. For pricing the call option $C$, we consider Gerber-Shiu’s option-pricing framework which can provide a unique price for the call option $C$ without imposing the stringent assumption of market completeness. For measuring risk of the call option $C$, there is no natural choice for the assignment of a probability distribution for the stock-price dynamic. In fact, even the existence of the probability distribution is controversial in philosophy, statistics and finance. For details, see Kyburg and Smokler (1964), Tong (1990), and Gerber and Shiu (2000) (in the discussion by Professor Phelim Boyle). It seems that the assignment of the probability distribution may depend on a statistical estimate and/or an investor’s subjective belief (or risk preference) (see Elliott and Kopp (1999)). Here, we consider the problem from the Bayesian viewpoint via the “Random Esscher Transform.” We introduce the concept of Bayesian Esscher “scenarios,” which further generalizes the concept of generalized “scenarios,” for the assignment of probability distributions (or generalized “scenarios”) which govern the stock-price movements. By modifying the representation form of coherent risk measures in Artzner et al. (1999), our risk measure consists of two distinctive parts. The first part involves the use of the risk-neutral Bayesian Esscher “scenario” for pricing while the second part involves the use of a family of real-world Bayesian Esscher “scenarios” for measuring risks. The risk measure is calculated as the supremum of the expected portfolio’s loss over the fixed time horizon with respect to the family of real-world Bayesian Esscher “scenarios.”

We present the main idea of our model as follows:

We fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P}_0)$, where $\mathbb{P}_0$ is a reference/control measure which is given/known in advance. Here, the sample space $(\Omega, \mathcal{F})$ represents the uncertainty of the stock-price movement. Let $\mathcal{T}$ denote the time index set of our financial model, where $\mathcal{T} \subseteq \mathbb{R}^+$, with $\mathbb{R}^+$ being the set of all non-negative real numbers. $\mathcal{T}$ is taken as $\mathbb{R}^+$ (or $\{0, 1, 2, \ldots \}$) in the continuous-time (or discrete-time) setting. Let $X_t$ be the continuously compounded rate of return of the risky underlying asset $S$ at time $t$, for each fixed $t \in \mathcal{T}$. Then, a stochastic process $\{X_t\}_{t \in \mathcal{T}}$ is a stochastic process defined on $(\Omega, \mathcal{F})$ with $X_0 = 0$ and taking values of the set on real line $\mathbb{R}$. We assume that the dynamic of the bond-price process $\{B_t\}_{t \in \mathcal{T}}$ and the stock-price process $\{S_t\}_{t \in \mathcal{T}}$ is governed by the following equations:

$$B_t = B_0 e^{rt}, \quad B_0 = 1,$$
\[ S_t = S_0 e^{X_t}, \quad S_0 = s, \quad t \in \mathcal{T}, \quad (2.1) \]

where \( r \) is the constant force of interest.

We equip our sample space \((\Omega, \mathcal{F})\) with the natural information structure \(\{\mathcal{F}_t^X\}_{t \in \mathcal{T}}\) generated by the values of the stock-price process \(\{X_t\}_{t \in \mathcal{T}}\). For each fixed \( t \in \mathcal{T} \), we can interpret \(\mathcal{F}_t^X\) as the information generated by the values of the stock-price process \(\{X_t\}_{t \in \mathcal{T}}\) up to time \( t \). Then, we assume that, under \( \mathbb{P}_0 \), \(\{X_t\}_{t \in \mathcal{T}}\) satisfies the following conditions:

1. \(\{X_t\}_{t \in \mathcal{T}}\) has stationary and independent increments.

2. For each \( t \in \mathcal{T} \), the random variable \(X_t\) has an infinitely divisible distribution \( F(x,t) \)

\[
F(x,t) = \mathbb{P}_0(\{\omega \in \Omega; X_t(\omega) \leq x\}), \quad \text{for each } x \in \mathbb{R}. \quad (2.2)
\]

In Gerber-Shiu’s option-pricing framework, the distribution function \( F(x,t) \) is assumed to be given/known in advance. Here, we also assume that \( F(x,t) \) is defined under the know reference probability \( \mathbb{P}_0 \). In order to obtain an objective pricing result, it may be more appropriate to start from an objective distribution \( F(x,t) \). In practice, we can fix an objective distribution \( F(x,t) \) according to a given set of historical data and assume that all agents agree on the same distribution \( F(x,t) \) as a reference distribution.

Let \( M_0(\theta, t) \) denote the moment generating function of the random variable \(X_t\) under \( \mathbb{P}_0 \). Note that \( M_0(\theta, t) \) exists if \( M_0(\theta, t) < \infty \) and that

\[
M_0(\theta, t) := E_{\mathbb{P}_0}(e^{\theta X_t}) = \int_{-\infty}^{\infty} e^{\theta x} dF(x,t), \quad t \in \mathcal{T}, \quad (2.3)
\]

where the integral is in the Riemann-Stieltjes sense.

Let \( \eta = \{\theta \in \mathbb{R}; M_0(\theta, t_0) < \infty, \text{for some } t_0 \in \mathcal{T} \setminus \{0\}\} \) be the set of Esscher parameters. From the definition of infinite divisibility, if \( M_0(\theta, t_0) \) exists for some

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1 Examples of infinitely divisible distributions include the normal distribution, the Poisson distribution, the gamma distribution and the inverse-Gaussian distribution, etc.
$t_0 \in T \setminus \{0\}$, $M_0(\theta, t)$ exists for all $t \in T$. For each $\theta \in \eta$, the Esscher measure $\mathbb{P}_\theta$ equivalent to $\mathbb{P}_0$ on $(\Omega, \mathcal{F})$ associated with the Esscher parameter $\theta$ is defined in such a way that, under $\mathbb{P}_\theta$, $\{X_t\}_{t \in T}$ satisfies the following conditions:

1. $\{X_t\}_{t \in T}$ has stationary and independent increments.
2. For each $t \in T$, the random variable $X_t$ has a distribution function

\[
F(x, t; \theta) = \frac{\int_{-\infty}^{x} e^{\theta y}dF(y, t)}{M_0(\theta, t)}. \tag{2.4}
\]

We denote $\mathcal{A}$ as the set of all subjective views (or risk preferences). For a subset $A$ of $\mathcal{A}$, we define a family of Bayesian Esscher “scenarios” $\{\mathcal{L}_a\}_{a \in A}$ associated with $A$ via the concept of “Random Esscher Transform.” Then, for each subjective view $a \in A$, we define a Bayesian Esscher “scenario” $\mathcal{L}_a$ as follows.

Let $\Theta$ be a random variable on the set $\eta$ such that, for each $\theta \in \eta$, $\Theta(\theta) = \theta$. Here, $\Theta$ represents a random Esscher parameter. Denote $\pi_a$ as the prior distribution of $\Theta$ associated with $a \in A$. That is, $\pi_a(\theta) = \pi_a(\{\Theta \leq \theta\})$. A random Esscher measure $\mathbb{P}_\Theta \sim \mathbb{P}_0$ on $(\Omega, \mathcal{F})$ associated with $\Theta$ is defined as a family of Esscher measures $\{\mathbb{P}_\theta \sim \mathbb{P}_0; \theta \in \eta\}$ such that, for each $\theta \in \eta$, $\mathbb{P}_\Theta$ is assigned as the Esscher measure $\mathbb{P}_\theta$. The random Esscher measure $\mathbb{P}_\Theta$ can be viewed as a particular case of a random measure equivalent to $\mathbb{P}_0$. Under $\mathbb{P}_\Theta$, the random distribution of $X_t$ is defined via the following “Random Esscher Transform” associated with the random Esscher parameter $\Theta$:

\[
F(x, t; \Theta) = \frac{\int_{-\infty}^{x} e^{\Theta y}dF(y, t)}{M_0(\Theta, t)}. \tag{2.5}
\]

Given $\Theta = \theta$, the random distribution (2.5) is assigned as $F(x, t, \theta)$. Let $\mathcal{N}$ be a $\sigma$-algebra of $\eta$, that is, $\mathcal{N}$ is the collection of all measurable subsets of $\eta$. Then, we define a product measure $\Pi_a$ on the space $(\Omega \times \eta, \mathcal{F} \otimes \mathcal{N})$ as follows:

For each $D \in \mathcal{F}$ and $N \in \mathcal{N}$,

\[
\Pi_a(D \times N) = \int_N \mathbb{P}_\theta(D)d\pi_a(\theta). \tag{2.6}
\]
Next, we define a Bayesian Esscher “scenario” $\mathcal{L}_a$ as the product space $(\Omega \times \eta, \mathcal{F} \otimes \mathcal{N}, \Pi_a)$ associated with $a \in A$. For the details about the theoretical background of Bayesian statistics, see Florens et al. (1990). On the product space $(\Omega \times \eta, \mathcal{F} \otimes \mathcal{N}, \Pi_a)$, the conditional process $\{X_t\}_{t \in T}|\{\Theta = \theta\}$ satisfies

1. $\{X_t\}_{t \in T}|\{\Theta = \theta\}$ has stationary and independent increments.

2. For each $t \in T$, the conditional distribution of $X_t$ given $\{\Theta = \theta\}$ is

$$F(x, t; \theta) = \mathbb{P}_\theta(\{\omega \in \Omega; X_t(\omega) \leq x\}).$$

(2.7)

In the finance literature, the “classical” scenario analysis refers to the assignment of a set of “scenarios” as a set of future contingencies (or possible outcomes). The set of “scenarios” can be chosen based on either an investor’s subjective view or historical data. The latter form of scenario analysis is known as stress testing which tests the effect on a portfolio under extreme outcomes (see Hull (2000) and Gleason (2000)).

The concept of generalized “scenarios” is the by-product of the representation form of coherent risk measures introduced by Artzner et al. (1999) and generalizes the “classical” scenario analysis by assigning a set of generalized “scenarios” as a set of probability measures on a given set of future contingencies. By choosing a set of probability measures as a set of Dirac measures on different future contingencies, we reduce the concept of generalized “scenarios” to the concept of classical “scenarios.” Besides providing a general methodology for assigning “scenarios,” the concept of generalized “scenarios” is essential for measuring risks of derivatives, especially when a valuation model is used under the no-arbitrage assumption. In the classical “scenarios” analysis, arbitrage opportunities may exist when the extreme “scenarios” (or outcomes) are considered (see Gleason (2000)). If we make a concession by considering the extreme “scenarios” probabilistically, arbitrage opportunities may be precluded. Hence, we may still preserve the no-arbitrage assumption by not assigning Dirac measures on the extreme “scenarios” within the framework of generalized “scenarios.” However, the concept of generalized “scenarios” still leaves a question on how a set of generalized “scenarios” should be chosen for measuring risk. In Artzner et al. (1999), Delbaen (1999) and Wang (1999), it has been pointed out that there is no agreement on the choice of generalized “scenarios” for measuring risk. It seems
that the choice of generalized “scenarios” is quite arbitrary or subjective. Here, the concept of Bayesian Esscher “scenarios” provides a method for choosing/assigning generalized “scenarios” by generalizing the concept of generalized “scenarios.” First, different sets of prior probabilities are assigned on a given set of generalized “scenarios” according to different subjective views or risk preferences. Then, by using Bayes formula\(^2\), we update the prior probabilities with observations about stock prices (or the market data) which provides a rather more scientific way in choosing generalized “scenarios.” Here, the Bayes formula decides how much weights should be placed on the market observations/data and a given subjective view/risk preference. Hence, we can relate the concept of Bayesian Esscher “scenarios” to the credibility theory by giving the credibility interpretation for the choice of generalized “scenarios” within the framework of Bayesian Esscher “scenarios.” The concept of Bayesian Esscher “scenarios” is flexible enough to incorporate both pricing framework and the framework for risk measurement. The risk-neutral Bayesian Esscher “scenario” is used for pricing derivatives while a family of real-world Bayesian Esscher “scenarios” are adopted for measuring risks of derivatives.

Now, we consider a portfolio of a single European call option \(C\) with strike price \(K\) and maturing at time \(T\), where \(T \in \mathcal{T}\). For the valuation of the portfolio \(C\), we adopt the risk-neutral Bayesian Esscher “scenario” which is equivalent to Gerber-Shiu’s pricing model. In the following, we construct the risk-neutral Bayesian Esscher “scenario” \(\mathcal{L}_q\) associated with the risk-neutral preference \(q \in \mathcal{A}\).

First, we impose the same assumptions on the financial market as in Gerber and Shiu (1994). We assume that the stock pays no dividends. Let \(\mathbb{P}_{\theta_q}\) be the risk-neutral Esscher measure associated with the risk-neutral parameter \(\theta_q\). In order to ensure the existence of \(\mathbb{P}_{\theta_q}\), we assume that both \(\theta_q\) and \(\theta_q + 1\) belong to the set \(\eta\). In Gerber and Shiu (1994), \(\mathbb{P}_{\theta_q}\) is determined in such a way that the discounted stock-price process \(\{e^{-rt}S_t\}_{t \in \mathcal{T}}\) is a \(\mathbb{P}_{\theta_q}\)-martingale. Hence, the parameter \(\theta_q\) is the solution of the following equation:

\[
S_0 = E_{\mathbb{P}_{\theta_q}}(e^{-rt}S_t). \tag{2.8}
\]

\(^2\)One may view “Random Esscher Transform” as a “random” Bayes formula which is used for changing probability measures in a random way. Hence, in our model, we use Bayes formula twice.
From (2.8), it can be shown that the parameter $\theta_q$ can be determined uniquely from the following equation (see Gerber and Shiu (1994)):

$$r = \ln[M_0(1, 1, \theta_q)],$$

(2.9)

where

$$M_0(1, 1, \theta_q) = \int_{-\infty}^{\infty} e^{x} dF(x, 1; \theta_q).$$

(2.10)

For the risk-neutral preference $q \in A$, we assume that the prior distribution $\pi_q$ of $\Theta$ is given by

$$\pi_q(\Theta = \theta_q) = 1.$$

(2.11)

Note that with the prior distribution $\pi_q$ being the point mass at $\theta_q$, the posterior distribution corresponding to $\pi_q$ remains the same as $\pi_q$ irrespective of the given information $\mathcal{F}_t^X$. The prior distribution $\pi_q$ reflects the strong prior/subjective belief on the use of risk-neutral Esscher measure $\mathbb{P}_{\theta_q}$ for pricing. In the language of credibility theory, this means that zero weight is placed on the market information/data in choosing the pricing probability measure under $\mathcal{L}_q$.

Then, the risk-neutral Esscher product measure $\Pi_q$ on $(\Omega \times \mathcal{F} \otimes \mathcal{N})$ with point mass marginal at $\theta_q$ on $\mathcal{N}$ is defined by

$$\Pi_q(F \times \{\theta_q\}) = \mathbb{P}_{\theta_q}(F), \quad \text{for each } F \in \mathcal{F}. \quad (2.12)$$

Intuitively, this means that, in the risk-neutral world, the subjective views and risk preferences do not play any role.

The risk-neutral Bayesian Esscher “scenario” $\mathcal{L}_q$ is defined as the product space $(\Omega \times \eta, \mathcal{F} \otimes \mathcal{N}, \Pi_q)$. Under $\mathcal{L}_q$, the value of the call option $C$ at time $t \in [0, T]$ is given by

$$C(S_t, t) = E_{\Pi_q}(e^{-r(T-t)} \max(S_t - K, 0) \mid \mathcal{F}_t^X).$$

(2.13)

The price of the call option $C$ is calculated as the expectation of the discounted payoff of $C$ at expiry with respect to the risk-neutral Esscher measure $\mathbb{P}_{\theta_q}$ under $\mathcal{L}_q$. 
Formula (2.13) is the same as Gerber-Shiu’s option-pricing formula. Gerber and Shiu (1994) provided the following analytical expression for the formula (2.13) holds.

\[
C(S_t, t) = S_t \left[ 1 - F(\ln\left(\frac{K}{S_t}\right), T - t, \theta_q + 1)\right] - e^{-r(T-t)} K \left[ 1 - F(\ln\left(\frac{K}{S_t}\right), T - t, \theta_q)\right].
\]  

(2.14)

Remarks:

1. The formula (2.14) is flexible enough to accommodate different parametric assumptions on the stock-price dynamic, namely the Wiener process, the Poisson process, the gamma process and the inverse-Gaussian process.

2. A common agreement on the value of the portfolio \( C \) is obtained if we assume the same starting distribution function \( F(x, t) \). This explains why we require the starting distribution \( F(x, t) \) to be determined objectively at the beginning of this section.

For measuring risk of the portfolio \( C \), we move from the risk-neutral world to the real world where, in contrast to the risk-neutral world, the investor’s subjective view (or risk preference) does play a vital role for the calculation of the risk measure. This motivates us to consider a family \( \{L_a\}_{a \in A} \) of real-world Bayesian Esscher “scenarios” associated with a set \( A \) of subjective views (or risk preferences) in the second part of our model.  

Let \([t, t+h]\) denote the fixed time horizon for risk measurement, where \( t, t+h \in [0, T] \). The time horizon \([t, t+h]\) may be chosen based on the liquidity of the portfolio, the decided holding period of the portfolio and the purpose of risk management, and

\(^3\)However, Ait-Sahalia and Lo (2000) suggested the use of the risk-neutral probability for the calculations of risk measures since they argued that an economic value should be placed on measuring the losses to a portfolio.
so on. Typically, it may be one day, one week, or one month according to different practical situations.

Now, given the price/market information up to time \( t \), say \( F^X_t \), we define the future-net-worth of the portfolio over \([t, t+h]\) as a random variable \( C(S_{t+h}, t+h) - e^{rh}C(S_t, t) \), denoted as \( \Delta C_{t,h} \), where both \( C(S_t, t) \) and \( C(S_{t+h}, t+h) \) can be obtained from the pricing formula (2.14). In the second part of our model, we calculate our risk measure as the supremum of the expectations of the discounted-future-net-loss \( e^{-rh}\Delta C_{t,h} \) with respect to the family of real-world Bayesian Esscher “scenarios” \( \{L_a\}_{a \in A} \). That is, given \( F^X_t \), we define the risk measure of the portfolio \( C \) over \([t, t+h]\) with respect to \( \{L_a\}_{a \in A} \) as follows:

\[
\rho_A(\Delta C_{t,h}|F^X_t) = \sup \{ -E\pi_a(e^{-rh}\Delta C_{t,h}|F^X_t)|a \in A\} .
\]  (2.15)

Our risk measure is defined from the viewpoint of a buyer. From a writer’s point of view, the future net loss of the call option becomes \( \Delta C_{t,h} \). This means that the future net loss from the writer’s viewpoint is the negative of the future net loss from the buyer’s perspective. It is interesting to note that our risk measure \( \rho_A \) provides some insights on how the three p’s of total risk management introduced by Lo (1999), namely price, probability and preference, interact with each other in measuring risks of derivatives.

We equip the product sample space \((\Omega \times \eta, \mathcal{F} \otimes \mathcal{N})\) with the enlarged information structure \( \{G_t^{\Theta, X}\}_{t \in T} \) generated by the values of \( \Theta \) and the stock-price process \( \{X_t\}_{t \in T} \). That is, for each \( t \in T \), \( G_t^{\Theta, X} \) represents the information generated by the values of \( \Theta \) and the stock-price process \( \{X_t\}_{t \in T} \) up to time \( t \). To calculate the risk measure (2.15), we first need to calculate the expectation \( E\pi_a(C(S_{t+h}, t+h)|F^X_t) \), for each \( a \in A \). By the tower law,

\[
E\pi_a(C(S_{t+h}, t+h)|F^X_t) = E\pi_a[E\pi_a(C(S_{t+h}, t+h)|G_t^{\Theta, X})|F^X_t] \\
= E\pi_a[E\pi_a(C(S_t e^{X_{t+h}-X_t}, t+h)|\Theta, F^X_t)|F^X_t] \\
= \int_{-\infty}^{\infty} E\pi_a(C(S_t e^{X_{t+h}-X_t}, t+h)|\Theta = \theta, F^X_t)d\pi_a(\theta|F^X_t)
\]
\[
\begin{align*}
&= \int_{-\infty}^{\infty} E_{\Pi_a}(C(S_t e^{X_h}, t + h)) d\pi_a(\theta|\mathcal{F}_t^X) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(S_t e^x, t + h) dF(x, h; \theta) d\pi_a(\theta|\mathcal{F}_t^X) \\
&\quad + C(S_t, t) | a \in A \\
&= \sup \left\{ -e^{-rh} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(S_t e^x, t + h) dF(x, h; \theta) d\pi_a(\theta|\mathcal{F}_t^X) \\
&\quad + C(S_t, t) | a \in A \right\} ,
\end{align*}
\]

(2.16)

Then, we can express the risk measure (2.15) in the integral form

\[
\rho_A(\Delta C_{t,h}|\mathcal{F}_t^X) = \sup \left\{ -e^{-rh} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(S_t e^x, t + h) dF(x, h; \theta) d\pi_a(\theta|\mathcal{F}_t^X) \\
&\quad + C(S_t, t) | a \in A \right\} ,
\]

(2.17)

where \( C(S_t e^x, t + h) \) is calculated from (2.14).

It is not difficult to check that the risk measure \( \rho_A \) satisfies the four coherent properties introduced by Artzner et al. (1999). The risk measure \( \rho_A \) depends on the given information structure \( \mathcal{F}_t^X \). In practice, \( \mathcal{F}_t^X \) is specified by market observations/data through discrete sampling, and in this case, the credibility interpretation for the posterior distribution \( \pi_a(\cdot|\mathcal{F}_t^X) \) becomes more transparent. We will calculate the posterior distribution \( \pi_a(\cdot|\mathcal{F}_t^X) \) explicitly via the Bayes formula for each special case in the next section.

Risk measures for portfolios of other European derivatives can be obtained in a similar fashion. From the practical viewpoint, it is convenient to set \( A \) as a finite set in order to simplify the calculation of the risk measure \( \rho_A \). Under some parametric assumptions on the stock-price dynamic, a closed-form expression for (2.17) can be obtained.

Remarks:

1. For each fixed \( a \in A \), \( E_{\Pi_a}(-e^{-rh} \Delta C_{t,h}|\mathcal{F}_t^X) \) is the best estimate of the discounted-future-net-loss \(-e^{-rh} \Delta C_{t,h}\) in the expected squared-loss-error sense with respect to the measure \( \Pi_a \). Hence, the risk measure \( \rho_A \) is just the best estimate of the loss \(-e^{-rh} \Delta C_{t,h}\) under the worst case “scenario” over \( A \). Actually, the concept of VaR and the scenario-based risk measure \( \rho_A \) are quite different. The former concerns the statistical estimation of the loss of a portfolio with certain degree of confidence (or probability level) while the latter deals with the estimation
of the loss of the portfolio under the worst-case “scenario.” Nevertheless, both risk measures express the concept of risk as the portfolio’s loss in domestic monetary units.

2. If $A_1 \subseteq A_2$, $\rho_{A_1}(\Delta C_{t,h} | \mathcal{F}_t^X) \leq \rho_{A_2}(\Delta C_{t,h} | \mathcal{F}_t^X)$. This means that the more “scenarios” you consider, the more conservative risk measures you obtain.

3. For measuring risk over a time horizon $[t, t+h]$, we suppose that the portfolio will not be liquidated or adjusted during the time interval $(t, t+h)$. However, we do not impose any restriction for the liquidity of the portfolio. Here, we consider risk in an non-cumulative sense over a time horizon $[t, t+h]$. Wang (1999) introduced a class of dynamic risk measures which can take the intermediate cash flows and position changes into account for evaluating risk in a cumulative sense.

4. In Artzner et al. (1997), it has been noted that VaR and the shortfall measure, which depends on a particular probability measure, cannot incorporate model risk. Here, we incorporate the model risk through the concept of Bayesian Esscher “scenarios” and the random Esscher parameter $\Theta$.

5. One may argue that it is not easy to implement the risk measure $\rho_A$ in some practical situations since the choice of the generalized “scenarios” involves human judgment or the practitioner’s subjective view (or risk preference). However, we contend that the role of human judgment provides an important way to improve risk measurement. Holton (1997) pointed out the subjective nature of risk and the inappropriateness of neglecting the role of human judgment for measuring risk. Also, in Artzner et al. (1997), it has been mentioned that the only way to improve risk management requires thinking before calculating risk measure.

6. We notice that $E_{\Pi_q}( - e^{-rh} \Delta C_{t,h} | \mathcal{F}_t^X ) = 0$. This implies that the risk-neutral Bayesian Esscher “scenario” can serve as a reference point for our risk measure. Suppose an agent chooses the set $A$ for calculating our risk measure. Then, the risk measure $\rho_A(\Delta C_{t,h} | \mathcal{F}_t^X)$ is positive (zero, negative) if and only if the agent is risk-averse (risk-neutral, risk-taking). This property plays a role similar to that of utility functions in financial economics.
7. If $\rho_A(\Delta C_{t,h}|\mathcal{F}_t^X) \leq 0$, the position $\Delta C_{t,h}$ over $[t, t+h]$ is said to be acceptable with respect to the set $A$ of subjective views. By translation invariance property, $\rho_A(\Delta C_{t,h} + e^{rh}\rho_A(\Delta C_{t,h}|\mathcal{F}_t^X)|\mathcal{F}_t^X) = 0$. This means that the risk measure $\rho_A$ can be interpreted as the minimum amount of capital invested in a reference or risk-free instrument over the time horizon $[t, t+h]$ in order to make the portfolio acceptable with respect to the scenario set $A$.

8. For regulators, they may use the risk measure $\rho_A$ as a benchmark for imposing margin requirements on different trading positions according to their subjective views (or risk preferences). For instance, if $\rho_A(\Delta C_{t,h}|\mathcal{F}_t^X) > 0$, it can be interpreted as the margin requirement that should be imposed in order to withstand the risk of the portfolio $\Delta C_{t,h}$. If $\rho_A(\Delta C_{t,h}|\mathcal{F}_t^X) < 0$, $-\rho_A(\Delta C_{t,h}|\mathcal{F}_t^X)$ can be interpreted as the cash amount that can be withdrawn from the current account so that one can still accept the portfolio $\Delta C_{t,h}$ with respect to the scenario set $A$.

9. The risk measure $\rho_A$ can be applied to an American call option written on a non-dividend-paying underlying stock. However, the application of $\rho_A$ for other complex American options requires further investigation. This constitutes an interesting topic for further research.

10. It is not difficult to extend the risk measure $\rho_A$ for dealing with a portfolio of European multi-state derivatives whose values are contingent on the stochastic movement of several underlying asset prices by following the same techniques in this section. Examples of multi-state derivatives include basket options, cross-currency options, exchange options, index options, and options on the extremum of several assets, etc. The extension of $\rho_A$ can also be applicable for insurance contracts which exhibit the nature of multi-state derivatives.

§3. Some special cases

We deal with some important special cases of our risk measures $\rho_A$ introduced in section two. First, we illustrate how to reduce our risk measure $\rho_A$ to a special case in which a set of generalized “scenarios” is specified by a family of subjective equivalent
Esscher measures associated with an index set of subjective Esscher parameters. In this case, the relationship between our risk measure \( \rho_A \) and the representation form of coherent risk measures becomes more clear. Then, we investigate the risk measure \( \rho_A \) under various parametric models for the dynamic of the stock-price process \( \{ S_t \}_{t \in T} \), such as the multiplicative binomial process, the Wiener process, the gamma process, the Poisson process, and the inverse-Gaussian process by applying some results in Gerber and Shiu (1994). Closed-form expressions for the risk measure \( \rho_A \) of a European call option can be obtained in some cases.

3.1 Subjective generalized “scenarios”

In this special case, the assignment of generalized “scenarios” becomes purely subjective since it does not take the market observations on stock prices into account. However, it provides some insights in understanding our framework of risk measurement. For instances, the relationship between our risk measure \( \rho_A \) and the set \( A \) of subjective views (or risk preferences) becomes more transparent by relating each Esscher measure to a subjective view (or risk preference) directly via the corresponding Esscher parameter. We can also relate our risk measure \( \rho_A \) to the VaR measure with certain confidence level by choosing an appropriate index set \( A \). Here, the choice of a family of subjective generalized “scenarios” plays a similar role as the confidence level in the VaR calculation. Although the use of subjective generalized “scenarios” is less “scientific” than the use of Bayesian Esscher “scenarios” for the calculation of the risk measure \( \rho_A \), it is convenient to use subjective generalized “scenarios” in order to obtain the closed-form expressions for the risk measure \( \rho_A \) in some special cases. In the following, we derive an analytical expression for the risk measure \( \rho_A \) under a family of subjective generalized “scenarios.”

Let \( A \) denote a sub-interval \([a_1, a_2]\) of \( \eta \). For each \( a \in A \), we assume that the prior distribution \( \pi_a \) of \( \Theta \) under \( L_a \) is defined by

\[
\pi_a(\{ \Theta = a \}) = 1.
\]

(3.1)

Then, the product measure \( \Pi_a \) on \((\Omega \times \eta, \mathcal{F} \otimes \mathcal{N})\) becomes the Esscher measure \( \mathbb{P}_\theta \) equivalent to \( \mathbb{P}_0 \) on \((\Omega, \mathcal{F})\) associated with the subjective parameter \( a \in A \). Hence,
the risk measure $\rho_A$ can be reduced to the following form:

$$\rho_A(\Delta C_{t,h}|F_t^X) = \sup \{-E_{\pi_a}(e^{-rh}\Delta C_{t,h}|F_t^X)|a \in [a_1, a_2]\}.$$  

(3.2)

We can also express the risk measure (3.2) in the following integral form:

$$\rho_A(\Delta C_{t,h}|F_t^X) = \sup \{-e^{-rh} \int_{-\infty}^{\infty} C(S_t e^{x}, t + h) dF(x, h; a) + C(S_t, t)|a \in [a_1, a_2]\}.$$  

(3.3)

For each fixed $a \in A$, the posterior distribution $\pi_a$ of $\Theta$ given $F_t^X$ remains the same as the prior distribution $\pi_a$ regardless of the given information $F_t^X$. This means that the choice of a generalized “scenario” is purely subjective for each $a \in A$. However, it should be noted that the case of subjective generalized “scenarios” and the no-data case in the credibility theory are two distinctive domains of specification. The former case assigns each generalized “scenario” as a degenerate random Esscher measure at the point $\{a\}$, for each subjective view $a \in A$, while, in the latter case, a non-degenerate prior distribution $\pi_a$ still assigns different probabilities on different points in the support of $\Theta$ under $L_a$ and the risk measure $\rho_A$ is defined with respect to the prior distribution as follows:

$$\rho_A(\Delta C_{0,h}|F_0^X) = \sup \{-e^{-rh} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(S_0 e^{x}, h) dF(x, h; \theta_a) d\pi_a(\theta_a) + C(S_0, 0)|a \in A\}.$$  

(3.4)

Finally, it is not difficult to observe that the risk measure (3.3) is of the same type as the representation form of coherent risk measures and that the risk measure (3.4) is a modification of the representation form of coherent risk measures by the random mixing parameter $\Theta$.

3.2 Multiplicative binomial model

The idea of using binomial model for pricing derivatives was suggested by the Nobel laureate W.F. Sharpe and developed in the paper by Cox, Ross and Rubinstein (CRR) (1979). From the theoretical point of view, CRR model provides insight into concept of risk-neutral valuation. From the practical viewpoint, it gives an efficient
and accurate approximation to the continuous-time option-pricing model. Inspired by
the spirit of CRR option-pricing model, we include the multiplicative binomial model
for measuring risk of the portfolio \( C \) here in order to provide a more transparent
way to present our framework for risk measurement and an easy way to implement
our risk measure \( \rho_A \). We point out that the multiplicative binomial model for the
stock-price dynamic is not infinitely divisible, and hence it may not be appropriate
to consider this subsection as a special case of section two. However, we employ the
same principle/technique as in section two, namely the concept of Bayesian Esscher
"scenarios," to measure the risk of the portfolio \( C \) under the multiplicative binomial
model.

First, we consider a two-level discrete-time binomial model. In the first level, we
fix a time index set \( T_k := \{0, 1, 2, \ldots, T\} \). For each \( k \in T_k \setminus \{T\} \), we suppose that risk
managers/traders are interested in measuring risks of derivatives over a one-period
time interval \([k, k+1]\). However, we do not impose any assumption on the liquidity of
derivatives over \([k, k+1]\). In the second level, we divide each time interval \([k, k+1]\) into
\( m \) sub-intervals of equal length and set the length of each interval as one basic unit.
Then, we fix a time index set \( T_n := \{0, 1, 2, \ldots, km, km + 1, \ldots, (k + 1)m, \ldots, Tm\} \)
for the basic time unit in the second level of the binomial model. The two-level
binomial model provides practitioners with the flexibility of adjusting \( m \) according
to their desirable degree of accuracy in the approximation to the continuous-time
model. It is a tailor-made model for the regular risk measurement practice. In the
following, we derive the closed-form expression of the risk measure \( \rho_A \) under the
two-level discrete-time binomial model.

First, we define a stochastic process \( \{Y_n\}_{n \in T_n} \) on the sample space \((\Omega, \mathcal{F})\) with
\( Y_0 = 0 \) and taking values on the set \( \{u, d\} \). Then, we assume that the bond-price
process \( \{B_n\}_{n \in T_n} \) and the stock-price process \( \{S_n\}_{n \in T_n} \) satisfy:

\[
B_{n+1} = B_n e^r, \quad B_0 = 1
\]

\[
Y_{n+1} = \ln \left( \frac{S_{n+1}}{S_n} \right), \quad S_0 = s, \quad \text{for each } n \in T_n \setminus \{Tm\},
\]

where \( S_n \) represents the stock price at the end of the \( n \)-th sub-interval. To preclude
arbitrage opportunities, we suppose that \( d < r < u \).
Let \( P_0 \) be the reference measure on \((\Omega, \mathcal{F})\). Then, we assume that, under \( P_0 \), \( \{Y_n\}_{n \in \mathcal{T}_n} \) satisfies the following conditions:

1. \( Y_1, Y_2, \ldots, Y_{T m} \) are independent and identically distributed.
2. \( P_0(\{Y_n = u\}) = p = 1 - P_0(\{Y_n = d\}). \)

Here, \( p \) is assumed to be given/known associated with \( P_0 \). Denote the set \( \{\theta \in \mathbb{R}; E_{P_0}(e^{\theta Y_n}) < \infty\} \) as \( \eta_Y \). Then, for each \( \theta \in \eta_Y \), the Esscher measure \( P_\theta \) equivalent to \( P_0 \) on \((\Omega, \mathcal{F})\) associated with \( \theta \) is defined in such a way that, under \( P_\theta \), \( \{Y_n\}_{n \in \mathcal{T}_n} \) satisfies the following conditions:

1. \( Y_1, \ldots, Y_{T m} \) are independent and identically distributed.
2. For each \( n \in \mathcal{T}_n \setminus \{0\} \), the random variable

\[
Y_n = \begin{cases} 
  u & \text{with probability } p(\theta) \\
  d & \text{with probability } 1 - p(\theta), 
\end{cases}
\]

where

\[
p(\theta) = \frac{e^{\theta u} p}{e^{\theta u} + e^{\theta d}(1 - p)}.
\]

Now, we consider a portfolio \( C \) of a single European call option with strike price \( K \), expiry at time \( T_m \) and written on \( S \). For pricing of the portfolio \( C \), we consider the risk-neutral Bayesian Esscher “scenario” \( L_q \). By following the same technique as in section two and using some results in Gerber and Shiu (1994), it is easy to show that, under \( L_q \), the value of the portfolio \( C \) at time \( n \in \mathcal{T}_n \) is given by:

\[
C_n = e^{-r(T_m-n)} \sum_{j=0}^{T_m-n} \binom{T_m-n}{j} p(\theta_q)^j (1 - p(\theta_q))^{T_m-n-j} \max(S_n u^j d^{T_m-n-j} - K, 0),
\]

where

\[
p(\theta_q) = \frac{e^r - e^d}{e^u - e^d}.
\]

Note that the pricing formula (3.7) is the same as the CRR option-pricing formula.
For measuring risk of the portfolio $C$ over $[k, k+1]$ with $k \in T_k \setminus \{T\}$, we consider a family of real-world Bayesian Esscher “scenarios” \{\mathcal{L}_a\}_{a \in \mathbb{A}} associated with the set $\mathbb{A}$ of subjective views. In practice, it is reasonable and convenient to work with a finite set $\mathbb{A}$. For each $a \in \mathbb{A}$, we consider a finite subset \{\theta_l\}_{l=1}^{q_a}$ of $\eta_Y$. Let $\Theta$ be the random Esscher parameter with prior distribution $\pi_a$ under $\mathcal{L}_a$. Then, we assume that $\pi_a$ is completely determined as follows:

$$
\pi_a(\{\Theta = \theta_{al}\}) = \pi_{al},
$$

where $\pi_{al} \geq 0$, $l = \{1, 2, \ldots, q_a\}$, and $\sum_{l=1}^{q_a} \pi_{al} = 1$, $a \in \mathbb{A}$.

For each $l \in \{1, 2, \ldots, q_a\}$, conditional on $\{\Theta_a = \theta_{al}\}$, \{\mathcal{Y}_n\}_{n \in \mathbb{T}_n}$ satisfies the two conditions in (3.6) with $\theta$ replaced by $\theta_{al}$. Let $\{\mathcal{F}_n^Y\}_{n \in \mathbb{T}_n}$ be the natural information structure generated by $\{\mathcal{Y}_n\}_{n \in \mathbb{T}_n}$. For each $n \in \mathbb{T}_n$, the set of observations $\{\mathcal{Y}_1, \ldots, \mathcal{Y}_n\}$ contains essentially the same amount of information as $\mathcal{F}_n^Y$. Let $I_{iu}$ be the indicator function of the event $\{\mathcal{Y}_i = u\}$. That is, $I_{iu} = 1$ if $\mathcal{Y}_i = u$; Otherwise, $I_{iu} = 0$. Then, by the Bayes formula, the posterior distribution $\pi_a$ of $\Theta$ given $\mathcal{F}_n^Y$ is specified as follows:

$$
\pi_a(\{\Theta = \theta_{al}\}|\mathcal{F}_n^Y) = \frac{p(\theta_{al}) \sum_{i=1}^{n} I_{iu} (1 - p(\theta_{al}))^{n - \sum_{i=1}^{n} I_{iu} \pi_{al}}}{\sum_{l=1}^{q_a} p(\theta_{al}) \sum_{i=1}^{n} I_{iu} (1 - p(\theta_{al}))^{n - \sum_{i=1}^{n} I_{iu} \pi_{al}}},
$$

for each $l = \{1, 2, \ldots, q_a\}$.

In the binomial model, the posterior distribution $\pi_a$ of $\Theta_a$ can be computed explicitly, and the credibility interpretation of our framework becomes more obvious. We can consider the conjugate-prior case by assuming $\pi_a$ as a beta distribution with parameter indexed by $a \in \mathbb{A}$. However, in this case, only analytical formula for the risk measure $\rho_A$ is obtained.

Given $\mathcal{F}_km$, denote $\Delta C_{k,m}$ as the random variable $C_{(k+1)m} - e^{mC_{km}}$ which represents the future-net-worth of the portfolio $C$ over $[k, k+1]$, $k \in T_k$. Then, by following the same procedure as in section two and using formula (3.7), we obtain the following closed-form expression of $\rho_A$ given $\mathcal{F}_km$:

$$
\rho_A(\Delta C_{k,m}|\mathcal{F}_km) = -e^{-r(T-k)m} \left\{ \sum_{l=1}^{q_a} \left( \sum_{i=1}^{m} \binom{m}{i} p(\theta_{al})^i (1 - p(\theta_{al}))^{m-i} \right) \right\}.
$$
\[
\left[ \sum_{j=0}^{(T-k-1)m} \binom{(T-k-1)m}{j} p(\theta_q)^j (1 - p(\theta_q))^{(T-k-1)m-j} \right. \\
\max \left( S_{km} u^{i+j} d^{(T-k)m-j-i} - K, 0 \right) \pi_{\tilde{a}}(\{\Theta = \theta_{\tilde{a}}\}|\mathcal{F}_{km}) \\
- \sum_{j=0}^{(T-k)m} \binom{(T-k)m}{j} p(\theta_q)^j (1 - p(\theta_q))^{(T-k)m-j} \\
\max \left( S_{km} u^i d^{(T-k)m-j} - K, 0 \right), \text{ for some } \tilde{a} \in A , \quad (3.11)
\]

where \( p(\theta_{\tilde{a}}) \) and \( p(\theta_q) \) are determined from (3.6) and (3.8), respectively.

Since \( A \) is a finite set, the worst-case “scenario” \( \tilde{a} \) can be identified directly in order to erase the supremum sign. It is interesting to note that the risk measure (3.11) can be modified to deal with other complex derivatives, such as American options and exotic options, because of the flexibility and simplicity of the binomial model.

### 3.3 Wiener Process

In this subsection, we deal with the Wiener case and obtain a closed-form expression for our risk measure. First, we let the time index set \( T \) be \([0, \infty)\). We assume that, under \( \mathbb{P}_0 \), \( \{X_t\}_{t \in [0, \infty)} \) is a Wiener process, that is

1. \( \{X_t\}_{t \in [0, \infty)} \) has stationary and independent increments.
2. For each \( t \in (0, \infty) \), the random variable \( X_t \) follows a normal distribution with mean \( \mu_0 t \) and variance \( \sigma_0^2 t \), where \( \mu_0 \) and \( \sigma_0^2 \) are given/known.

For pricing the portfolio \( C \), we consider the risk-neutral Bayesian Esscher “scenario.” By following the same procedure as in section two and some results in Gerber and Shiu (1994), it can be shown that the pricing formula (2.14) coincides with the celebrated Black-Scholes formula under the assumption that the stock prices are log-normally distributed. Hence, we can apply the celebrated Black-Schole’s formula with known volatility \( \sigma_0 \) to calculate the market values of the portfolio \( C \) at time \( t \)
and time \( t + h \). Although Gerber-Shiu’s option-pricing formula coincides with the celebrated Black-Schole’s formula under the log-normality assumption for the stock-price dynamic, its derivation does not require the stringent assumption of market completeness. In the following, we derive the closed-form expression of \( \rho_A \) by considering the normal-normal-conjugate-prior case and a finite set \( A \) of subjective views (or risk preferences).

Suppose the prior distribution \( \pi_a \) of \( \Theta \) is a normal distribution with known mean \( m_a \) and variance \( \tau_a \), associated with each \( a \in A \). Then, we assume that, conditional on \( \{ \Theta = \theta \} \}, \{ X_t \}_{t \in [0, \infty)} \) is a Wiener process with mean \( \mu_0 + \theta \sigma_0^2 \) and variance \( \sigma_0^2 \) per unit time. In practice, the stock-price dynamic is observed via discrete sampling. For simplicity, we consider the discrete sampling with equal length of window and set the length of each sampling interval to be one unit of time. Let \( h \) and \( T \) be two positive integers. Denote \( Z_j \) as the random variable \( X_j - X_{j-1} \), for each \( j \in \{1, 2, \ldots, T\} \). Then, conditional on \( \{ \Theta = \theta \} \}, \{ Z_1, \ldots, Z_T \} \) are independent and identically distributed with common distribution being the normal distribution with mean \( \mu_0 + \theta \sigma_0^2 \) and variance \( \sigma_0^2 \). Given \( F_t^X \) (with positive integer \( t \)), the set of observations \( \{ Z_1, \ldots, Z_t \} \) is given/known. In order to obtain the posterior distribution of \( \Theta \) given \( F_t^X \) under the discrete-sampling setting, we only need the information on the set of observations \( \{ Z_1, \ldots, Z_t \} \). From some standard calculations, it is not difficult to show that the posterior distribution of \( \Theta \) given \( \{ Z_1, \ldots, Z_t \} \) is again a normal distribution with mean \( m_a \| t := (\frac{1}{\sigma_0^2 \tau_a} + t)^{-1}[\frac{1}{\sigma_0^2 \tau_a} (\mu_0 + m_a \sigma_0^2) + \frac{1}{\sigma_0^2} \bar{Z}] - \frac{m_\|}{\sigma_0^2} \) and variance \( \tau_a \| t := (\frac{1}{\tau_a} + t \sigma_0^2)^{-1} \), where \( \bar{Z} \) is the sample mean of \( \{ Z_1, \ldots, Z_t \} \). The credibility interpretation of the posterior distribution of \( \Theta \) becomes clear in the normal-normal-conjugate-prior case.

Given \( \{ Z_1, \ldots, Z_t \} \), we obtain the closed-form expression of \( \rho_A \) over the time horizon \( [t, t + h] \) as follows:

\[
\rho_A(\Delta C_{t,h}|\{ Z_1, \ldots, Z_t \}) = S_t \left[ \Phi(d_1) - e^{(\mu_0 + 1/2 \sigma_0^2 + m_a t \sigma_0^2 + 1/2 \tau_a t \sigma_0^2)h} \cdot \Phi \left( \frac{D_{a1}}{\sqrt{1 + D_{a2}^2}} \right) \right] \\
- K e^{-r(T-t)} \left[ \Phi(d_2) - \Phi \left( \frac{D_{a3}}{\sqrt{1 + D_{a2}^2}} \right) \right],
\]

where

\[
d_1 = \frac{\ln \left( \frac{S_t}{K} \right) + (r + \frac{1}{2} \sigma_0^2)(T - t)}{\sigma_0 \sqrt{T - t}},
\]

and

\[
D_{a1} = \sigma_0 \sqrt{T - t}, \quad D_{a3} = \sigma_0 \sqrt{T - t}.
\]
\[ d_2 = d_1 - \sigma_0 \sqrt{T - t}, \]
\[ D_{a1} = \frac{\ln(\frac{S_t}{K}) + (\mu_0 + \frac{1}{2} \sigma_0^2 - r)h + (r + \frac{1}{2} \sigma_0^2)(T - t) + (m_{a|t} + \tau_{a|t} \sigma_0^2 h)\sigma_0^2 h}{\sigma_0 \sqrt{T - t}}, \]
\[ D_{a2} = \frac{\sigma_0 h \sqrt{\tau_{a|t}}}{\sqrt{T - t}}, \]
\[ D_{a3} = \frac{\ln(\frac{S_t}{K}) + (\mu_0 + \frac{1}{2} \sigma_0^2 - r)h + (r - \frac{1}{2} \sigma_0^2)(T - t) + m_{a|t} \sigma_0^2 h}{\sigma_0 \sqrt{T - t}}. \]

Note that the worst-case “scenario” \( \tilde{a} \in A \) can be identified easily since \( A \) is a finite set. The parameters \( \mu_0 \) and \( \sigma_0^2 \) can be conveniently set as the sample mean and variance of \( \{Z_1, \ldots, Z_t\} \), respectively. The closed-form expression (3.12) generalizes the risk measure for a standard call option in Yang and Siu (2001). Finally, it is worth noting that a Bayesian Esscher “scenario” in the Wiener case resembles a Bayesian location mixture of normal models.

3.4 Other processes

We derive an analytical expression for the risk measure \( \rho_A \) in the case of shifted gamma process. The cases of shifted Poisson and inverse-Gaussian processes can be considered in a similar way. First, let the time index set \( T \) be \([0, \infty)\). Then, we assume that, under \( \mathbb{P}_0 \), \( \{X_t\}_{t \in [0, \infty)} \) satisfies:

1. \( \{X_t\}_{t \in [0, \infty)} \) has stationary and independent increments.

2. For each \( t \in [0, \infty) \), the random variable \( X_t \) has distribution function

\[
F(x, t) = \int_0^{x+c_0 t} G(y; \alpha_0 t, \beta_0)dy, \tag{3.13}
\]

where the parameters \( \alpha_0, \beta_0 \) and \( c_0 \) are given/known associated with \( \mathbb{P}_0 \) and \( G(y; \alpha, \beta) \) is the density function of the gamma distribution with shape parameter \( \alpha \) and scale parameter \( \beta \). In practice, the parameters \( \alpha_0, \beta_0 \) and \( c_0 \) can be determined by the method of moments (see Gerber and Shiu (1994)).

By considering the risk-neutral Bayesian Esscher “scenario” and using some results in Gerber and Shiu (1994), the value of the portfolio \( C \) at time \( t \in [0, T] \) is
given by:

\[
C^G(S_t, t) = S_t \left[ 1 - \int_0^{\ln(S_T/T) + \alpha_0(T-t)} G(y; \alpha_0(T-t), \beta_q - 1) dy \right]
- Ke^{-(T-t)} \left[ 1 - \int_0^{\ln(S_T/T) + \alpha_0(T-t)} G(y; \alpha_0(T-t), \beta_q) dy \right],
\]

(3.14)

where \( \beta_q \) is determined as

\[
\beta_q = \frac{1}{1 - e^{-(\alpha_0 + r)/\alpha_0}}.
\]

(3.15)

The pricing formula (3.14) does not depend on the parameter \( \beta_0 \).

In order to simplify the final expression of our risk measure, it is convenient to assume \( A \) to be a finite set.

For each \( a \in A \), we consider a finite subset \( \{ \theta_a_1, \ldots, \theta_a_{q_a} \} \) of \( \eta \). Let \( \Theta \) be the random Esscher parameter with prior distribution \( \pi_{a_l} \). Then, we assume that \( \pi_{a_l} \) is specified as in (3.9).

For each \( l \in \{1, 2, \ldots, q_a \} \), conditional on \( \{ \Theta = \theta_{a_l} \} \), \( \{ X_t \}_{t \in [0, \infty)} \) satisfies the following conditions:

1. \( \{ X_t \}_{t \in [0, \infty)} \mid \{ \Theta = \theta_{a_l} \} \) has stationary and independent increments.

2. For each \( t \in [0, \infty) \), the conditional distribution of the random variable \( X_t \) given \( \{ \Theta = \theta_{a_l} \} \)

\[
F(x, t; \theta_{a_l}) = \int_0^{x + \alpha_0 t} G(y; \alpha_0 t, \beta_0 - \theta_{a_l}) dy.
\]

(3.16)

As in the previous subsection, let \( t, h \) and \( T \) be positive integers. \( Z_j \) denotes the random variable \( X_j - X_{j-1} \), for each \( j \in \{1, 2, \ldots, T \} \). Then, conditional on \( \{ \Theta = \theta_{a_l} \} \), \( \{ Z_1, \ldots, Z_T \} \) are independent and identically distributed with common distribution \( F(x, 1; \theta_{a_l}) \), for each \( l \in \{1, 2, \ldots, q_a \} \). Then, the posterior distribution of \( \Theta \) given \( \{ Z_1 = z_1, \ldots, Z_t = z_t \} \) is given by:
\[
\pi_a(\{\Theta = \theta_{al}\}|\{Z_1 = z_1, \ldots, Z_t = z_t\}) = \frac{\prod_{i=1}^{t} G(z_i + c_0; \alpha_0, \beta_0 - \theta_{al})\pi_{al}}{\prod_{i=1}^{q_a} \prod_{l=1}^{g_c} G(z_i + c_0; \alpha_0, \beta_0 - \theta_{al})\pi_{al}},
\]
for each \(l \in \{1, 2, \ldots, q_a\}\).

Given \(\{Z_1 = z_1, \ldots, Z_t = z_t\}\), we obtain the following analytical expression of \(\rho_A\) over the time horizon \([t, t+h]\):

\[
\rho_A(\Delta C_{t,h}|\{Z_1 = z_1, \ldots, Z_t = z_t\}) = -e^{-rh} \left\{ \sum_{l=1}^{q_a} \left[ \int_{0}^{\infty} C^G(S_te^{y-ct}, t+h) \cdot G(y; \alpha_0t, \beta_0 - \theta_{al})dy \right] \pi_{al}(\{\Theta = \theta_{al}\}|\{Z_1 = z_1, \ldots, Z_t = z_t\}) + C^G(S_t, t) \right\},
\]
where \(C^G\) is calculated from (3.14) and the worst-case “scenario” \(\bar{a}\) erases the supremum sign.

§4. Conclusion and Further Research

We have proposed a new model for measuring risks of derivatives which is easy to implement and is flexible enough to preserve the coherence for measuring risks of a wide variety of derivatives. Through the use of Gerber-Shiu’s option-pricing framework, our approach provides investors with more flexibility in measuring risks of derivatives under several parametric models for the stock-price dynamic. A new concept, called the Bayesian Esscher “scenarios,” which generalizes the concept of generalized “scenarios,” has been introduced via “Random Esscher Transform” in order to take both the subjective views (or risks preferences) and the market observations (or data) into account in assigning generalized “scenarios.” We do not claim
that our risk measure can replace other risk measures in the literature for measuring risks of derivatives. In fact, none of the risk measures in the literature are perfect and a good risk management practice requires a good combination of all techniques in risk measurement, such as VaR, stress-test, scenario analysis, etc. This somehow suggests that risk management is as much an art as it is a science. Our aim is to provide an example for actuaries and risk researchers on how a time-honored tool in actuarial science, namely Esscher transform, can be used for risks measurement.

One possible extension to our model is the use of our framework for measuring risks of a portfolio of complex derivatives, such as American options and exotic options, etc. The investigation of the relationship between the concept of Bayesian Esscher “scenarios” and the Extreme-Value-Theory (EVT) represents an interesting topic for further research. For the details of EVT, see Embrechts et al. (1997). The parametric models for the stock-price dynamic included in this paper are all positively skewed. However, it may happen that the real stock-price data is negatively skewed. This motivates the development of a negatively skewed and infinitely divisible parametric model for the stock-price dynamic in future research. In the cases of American and exotic options, we may adopt the P.D.E. approach introduced by Siu and Yang (2000) for calculating our risk measure. A difficult and challenging topic for further research is the extension of our model to a non-linear time series framework because of the complexity of the pricing model under a non-linear time series framework. Roger and Satchell (1996) and Tong (1990) may provide some insights for tackling this difficult problem.

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